Sequential Synchronization of Chaotic Systems with an Application to Communication

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We propose a hierarchically structured communication system by using sequentially synchronized chaotic systems. Sequential synchronization is attained by first feeding a noiselike signal to a variable of the first transmitter and its receiver simultaneously and then feeding a variable of the first transmitter and its receiver to a variable of the second transmitter and its receiver, respectively, for subsequent feedings of variables in sequence. When this is applied to communication, the hierarchical structure enables selective protection of information due to the sequential property. We illustrate this in sequentially synchronized Navier-Stokes and Lorenz equations.

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Recently secure communication using chaos synchronization has attracted much attention as one of the important practical applications of chaos. The main idea is that when an information signal is masked by a large amplitude of chaotic signal, it can be easily recovered at the receiver by means of chaos synchronization, while external attack is almost impossible during transmission. Since the first report on chaos synchronization by Pecora and Carroll [1], and the following circuit implementation for communication by Cuomo and Oppenheim [2], many communication methods using chaos synchronization have been developed [3,4] and demonstrated in electronic circuits [2,5] and optical systems [6]. As for the efficiency of communication using chaos, there also have been many considerations in a technical respect [7].

In communication, it sometimes happens that one needs a system which allows privileged users to access information of a higher level of importance, while general users can access information of modest importance. By using a transmission system with a hierarchical structure, one can hope to protect the security of communication selectively, while not hampering the general users' access. The motivation of our study is to address this question of how to make a hierarchically structured communication system using chaos synchronization. As the answer, in this Letter we propose a sequential synchronization scheme that is applicable to a hierarchically structured communication system.

The main procedure of the sequential synchronization method is that we first feed a common, arbitrary noiselike signal to a variable of the first transmitter and its receiver chaotic system simultaneously. Next, we feed a variable of the first transmitter and its receiver to a variable of the second transmitter and its receiver, respectively. And then, we feed a variable of the second transmitter and its receiver to a variable of the third transmitter and its receiver, and so forth. When this synchronization scheme is applied, the communication system can have a hierarchical structure because of the sequential property. So a lower level user who has only the first chaotic system may access an information signal masked by the dynamics of the first chaotic system but cannot access an information signal masked by the signal of the second chaotic system. Additionally, the attractor of each chaotic signal is so strongly modified as to make the system unidentifiable even by the predictive modeling and noise reduction methods [8]. We explain the method of sequential synchronization and its application to communication with emphasis on the attainment of a hierarchical structure. We illustrate the profitable characteristics of the communication in sequentially synchronized five-dimensional Navier-Stokes and three-dimensional Lorenz equations.

The sequential synchronization scheme is as follows:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{F}_1(x_1, x_2, \ldots, [\alpha_1 x_i + \beta_1 f(t)], \ldots),\\
\dot{\mathbf{y}} &= \mathbf{F}_2(y_1, y_2, \ldots, [\alpha_2 y_j + \beta_2 x_l], \ldots),\\
\dot{\mathbf{z}} &= \mathbf{F}_3(z_1, z_2, \ldots, [\alpha_3 z_k + \beta_3 y_m], \ldots),\\
&\cdots \quad \text{(transmitter chaotic systems)},\\
\dot{\mathbf{x}}' &= \mathbf{F}_1(x'_1, x'_2, \ldots, [\alpha_1 x'_i + \beta_1 f(t)], \ldots),\\
\dot{\mathbf{y}}' &= \mathbf{F}_2(y'_1, y'_2, \ldots, [\alpha_2 y'_j + \beta_2 x'_l], \ldots),\\
\dot{\mathbf{z}}' &= \mathbf{F}_3(z'_1, z'_2, \ldots, [\alpha_3 z'_k + \beta_3 y'_m], \ldots),\\
&\cdots \quad \text{(receiver chaotic systems)},
\end{aligned}
\tag{1}
$$

where α_i and β_i are the scaling values for coupling. In the first pair of chaotic systems, the arbitrary common $f(t)$ is fed to x_i and x'_i simultaneously. In the second pair, the signals x_l and x'_l are fed to y_j and y'_j, respectively. If the first pair is synchronized by $f(t)$, $x_l = x'_l$, then the second pair can be synchronized by x_l and x'_l. The succeeding pairs of chaotic systems are synchronized in the same way.

Sequential synchronization is a modification of the type of chaos synchronization attained by feeding an external common noise [9,10]. So, to explain its way of working, we consider two logistic maps driven by the random noise $\beta\xi_n$ and $\beta\xi'_n$, respectively, whose equations are

$$
\begin{aligned}
x_{n+1} &= \lambda x_n(1 - x_n) + \beta\xi_n,\\
x'_{n+1} &= \lambda x'_n(1 - x'_n) + \beta\xi'_n.
\end{aligned}
\tag{2}
$$

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Through linear transformations, $x_n = \alpha y_n + \beta \xi_{n-1}$ and $\lambda/\alpha = \Lambda$, we obtain the following equations:

$$\begin{aligned} y_{n+1} &= \Lambda(\alpha y_n + \beta \xi_{n-1})[1 - (\alpha y_n + \beta \xi_{n-1})], \\ y'_{n+1} &= \Lambda(\alpha y'_n + \beta \xi'_{n-1})[1 - (\alpha y'_n + \beta \xi'_{n-1})]. \end{aligned} \tag{3}$$

Then ξ_{n-1} and ξ'_{n-1} are fed to y_n and y'_n, respectively, with the scaling factors α and β, the same as in Eq. (1).

If we assume that ξ_{n-1} and ξ'_{n-1} are synchronized signals of the first chaotic systems and that Eq. (3) represents the second chaotic systems, we can obtain the synchronization error of Eq. (3) such that $z_{n+1} = u_n z_n + A z_n^2$ by letting $\xi_{n-1} = \xi'_{n-1}$, where $z_n = y_n - y'_n$, $u_n = \Lambda\alpha(1 - 2\beta\xi_n - 2\alpha x_{n-1})$, and $A = \lambda\alpha^2$. The dynamics of z_n corresponds to the transverse motion from the synchronization manifold [11]. The above equation can be written as $z_{n+1} = \exp(n\langle \ln u_n \rangle) z_1$, where $\langle \ln u_n \rangle = \int \ln(u) P(u)\, du$ and $P(u)$ is the normalized probability distribution of u_n. Since $\langle \ln u_n \rangle$ is the transverse Lypunov exponent, z_n can converge to 0 when $\langle \ln u_n \rangle$ is negative [10–12]. That is, the two logistic maps are weakly synchronized [13]. So we can easily calculate the condition of α and β for synchronization [4,10,11]. Figure 1 is the temporal behaviors of $y_n - y'_n$ for $\Lambda = 3.8$. When $\alpha = 0.8$ and $\beta = 0.2$, $y_n - y'_n \to 0$, as shown in Fig. 1(a). However, when $\alpha = 0.85$ and $\beta = 0.15$, intermittent desynchronization appears as shown in Fig. 1(b) [12]. From this, we can understand that there exists a synchronization region [4,10,14].

The characteristics of sequential synchronization are studied in the following two pairs of chaotic systems:

$$\begin{aligned} \dot{x}_i &= -1.9x_i + 4[\alpha_1 y_i + \beta_1 f(t)]z_i + 4u_i v_i, \\ \dot{y}_i &= -7.2[\alpha_1 y_i + \beta_1 f(t)] + 3.2x_i z_i, \\ \dot{z}_i &= -4.7z_i - 7.0x_i[\alpha_1 y_i + \beta_1 f(t)] + k, \\ \dot{u}_i &= -5.3u_i - x_i v_i, \\ \dot{v}_i &= -v_i - 3.0x_i u_i \quad \text{(Navier-Stokes)}, \end{aligned} \tag{4}$$

$$\begin{aligned} \dot{p}_i &= \sigma[(\alpha_2 q_i + \beta_2 z_i) - p_i], \\ \dot{q}_i &= c p_i - (\alpha_2 q_i + \beta_2 z_i) - p_i r_i, \\ \dot{r}_i &= p_i(\alpha_2 q_i + \beta_2 z_i) - b r_i \quad \text{(Lorenz)}, \end{aligned} \tag{5}$$

where the subscript i is the transmitter or the receiver system when $i = 1$ or $i = 2$, respectively; k, σ, c, and b are the parameters which are taken to be 31.5, 10.0, 28.0, and 8/3, respectively; and α_i and β_i are the scaling factors for coupling. The first pair is the five-dimensional Navier-Stokes equations where $f(t)$ is fed to y_1 and y_2 simultaneously, and the second is the Lorenz systems where $z_1(t)$ and $z_2(t)$ are fed to q_1 and q_2, respectively. In the calculation, we let $f(t) = A\sin(2\pi\nu t)$ and change the amplitude and frequency at each period of $f(t)$ by using $A = 50\xi$ and $\nu = 0.8(\frac{1}{2} + \xi')$. Here ξ and ξ' are pseudorandom numbers in the range ξ and $\xi' \in (0, 1)$.

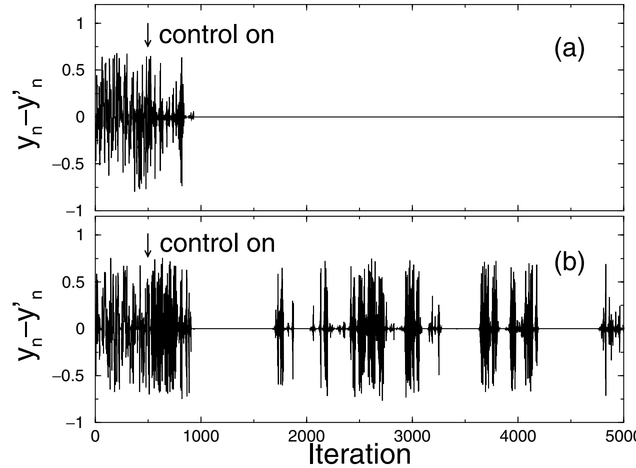

FIG. 1. Temporal behaviors of the difference motion of the two logistic maps when they are (a) synchronized at $\alpha = 0.8$ and $\beta = 0.2$, and (b) intermittently synchronized at $\alpha = 0.85$ and $\beta = 0.15$.

The traces in Figs. 2(a)–2(f) show the temporal behaviors of $f(t)$, $x_1(t)$, $x_1(t) - x_2(t)$, $z_1(t)$, $r_1(t)$, and $r_1(t) - r_2(t)$, respectively, when $\alpha_1 = 1.2$, $\beta_1 = 0.9$, $\alpha_2 = 0.9$, and $\beta_2 = 22.5$. As shown in Figs. 2(c) and 2(f), $x_1 - x_2$ and $r_1 - r_2$ converge to zero rapidly as time evolves. This means that the Navier-Stokes equations are synchronized by $f(t)$, whereas the Lorenz equations are synchronized by $z_1(t)$ and $z_2(t)$. Here, the range of α_1 and β_1 for synchronization of the Navier-Stokes equations depends upon A

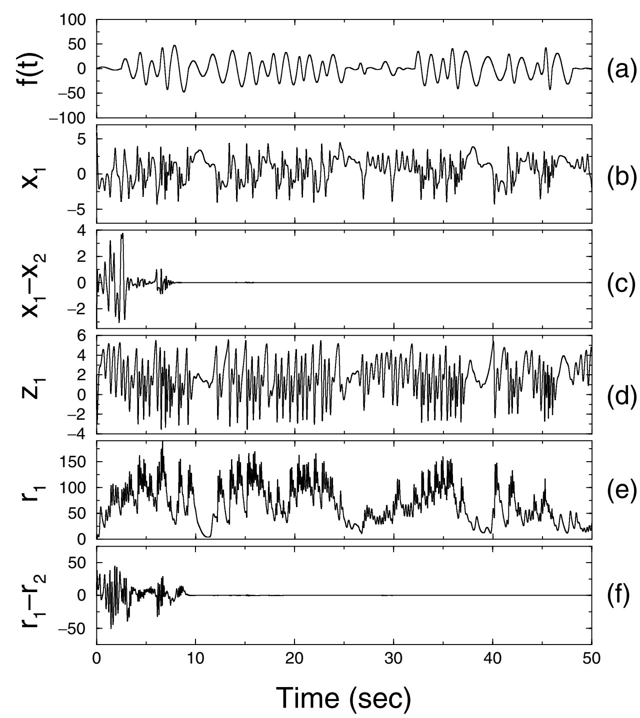

FIG. 2. Temporal behaviors of the control signals, the transmitting signals, and the difference motion of the variables when synchronization occurs: (a) $f(t)$, (b) $x_1(t)$, (c) $x_1(t) - x_2(t)$, (d) $z_1(t)$, (e) $r_1(t)$, and (f) $r_1(t) - r_2(t)$.

and ν of $f(t)$. On the other hand, the range of α_2 and β_2 for synchronization of the Lorenz equations depends upon $f(t)$, α_1, and β_1. For another example of sequential synchronization, we tested up to ten pairs of different chaotic systems, such as Rössler → Lorenz → Navier-Stokes → Duffing → forced Brusselator → Van der Pol → Lorenz → Navier-Stokes → Rössler → Duffing equations, and, as a result, found that the time required for synchronizing each pair depends on that of the prior chaotic system as well as on the scaling factors for coupling.

What interests us in Fig. 2 is the temporal behaviors of $x_1(t)$ and $r_1(t)$, which are strongly modified by $f(t)$ and $z_1(t)$. Insofar as we transmit only $f(t)$, $x_1(t)$, and $r_1(t)$ in communication, we can construct the attractors in such phase spaces as $f(t)$ versus $x_1(t)$, $f(t)$ versus $r_1(t)$, $x_1(t)$ versus $x_1(t + \tau)$, and $r_1(t)$ versus $r_1(t + \tau)$, where $\tau = 0.125$, as shown in Figs. 3(a), 3(b), 3(c), and 3(d), respectively. Especially, the attractors of the Navier-Stokes [Fig. 3(c)] and the Lorenz equations [Fig. 3(d)] in time-delayed coordinates do not leave any traces of their own original structures. So we can understand that it is hard to reconstruct the original attractors of the chaotic systems and to estimate the number of parameters even in a low-dimensional system because of the strong modification.

We obtained the correlations of the transmitting signals in order to show the deformation of the transmitting signals quantitatively. Figures 4(a), 4(b), and 4(c) are the autocorrelations of $f(t)$, $x_1(t)$, and $r_1(t)$, respectively. Figures 4(d), 4(e), and 4(f) are the cross correlations of $f(t)$ versus $x_1(t)$, $f(t)$ versus $r_1(t)$, and $x_1(t)$ versus $r_1(t)$, respectively. The autocorrelation of $f(t)$, of which the correlation time is about 1 sec, is given as an example of a noiselike signal that is made by a δ-correlated random number. When we compare the autocorrelations of $x_1(t)$ and $r_1(t)$ with the correlation function in Fig. 4(a), they have a short correlation time of about 1 sec and do not exhibit any periodic structure in their lag dependence. From this result, we understand that $x_1(t)$ and $r_1(t)$ act as noiselike signals. The cross correlations in Figs. 4(d), 4(e), and 4(f) also show a short correlation time which indicates that the signals are uncorrelated with each other.

Owing to the deformation and short correlation time of the transmitting signals, the communication system using this method has a number of merits with respect to security [8]. (i) The temporal behaviors of the transmitting signals are so strongly modified and uncorrelated. (ii) The second chaotic systems of the Lorenz equations are synchronized without transmitting $z_1(t)$ or $z_2(t)$ of the first chaotic systems. (iii) We can transmit an arbitrary real noise signal instead of $f(t)$. In this case we can recover the same $f(t)$ at the transmitter and the receiver system, by using identical bandpass filters.

The schematic diagram of the communication using sequential synchronization is shown in Fig. 5. In this system, an information signal can be masked by y or v and transmitted to the receiver through channels n_1 or n_2, respectively. The most significant characteristic of this communication system is the hierarchical structure that to our knowledge has never been suggested in any other secure communication system. To emphasize, in our communication system, while an information signal transmitted through an n_1 channel can be accessed by the general user who has the chaotic system A, another signal transmitted through an n_2 channel can be accessed only by a special user who has two chaotic systems, A and B, in simultaneous operation. Similarly, by transmitting a top secret

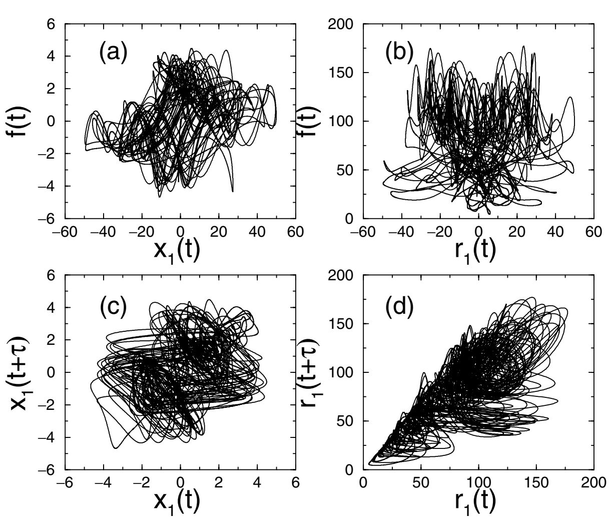

FIG. 3. Phase diagrams of the transmitting signals for $50.0 < t < 100.0$ when the transmitter and its receiver system are synchronized sequentially. (a) $f(t)$ versus $x_1(t)$, (b) $f(t)$ versus $r_1(t)$, (c) $x_1(t)$ versus $x_1(t + \tau)$, and (d) $r_1(t)$ versus $r_1(t + \tau)$, where $\tau = 0.125$.

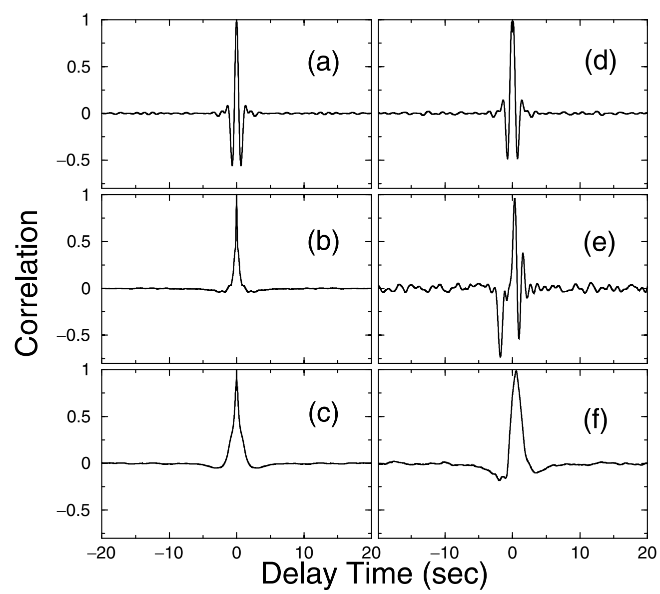

FIG. 4. Correlations of the transmitting signals: autocorrelations of (a) $f(t)$, (b) $x_1(t)$, and (c) $r_1(t)$, and cross correlation of (d) $f(t)$ versus $x_1(t)$, (e) $f(t)$ versus $r_1(t)$, and (f) $x_1(t)$ versus $r_1(t)$.

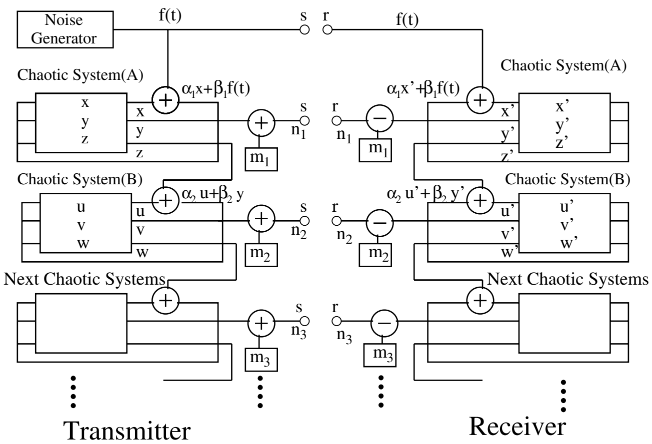

FIG. 5. Schematic diagram of secure communication using sequential synchronization: m_i are the information signals, n_i are the transmitting signals, s are the sending terminals, and r are the receiving terminals.

information signal through the last channel, we can make certain that only the most privileged group of users can access it. So, according to the level of secrecy required, we can control the degree of security.

In conclusion, we have proposed a sequential synchronization method applicable to communication. As a result the communication system enables selective protection of information for its hierarchical structure and deforms the transmitting signals so seriously as to make the system unidentifiable.

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