Simple Theory for the Two-Dimensional Child-Langmuir Law

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This paper presents, for the first time, a simple analytic theory for the two-dimensional (2D) Child-Langmuir law. For electron emission over a finite patch on a planar cathode, the limiting current density is derived approximately from first principles. The scaling laws are in excellent agreement with simulation results. They predict the onset of virtual cathode formation in a 2D geometry; they also indicate that electrons emitted from a cathode over only a restricted area may have a current density much exceeding the classical (1D) Child-Langmuir value.

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The Child-Langmuir Law [1–3] gives the maximum current density that can be transported across a planar gap of gap separation D and gap voltage V. In the one-dimensional model, this maximum current density is given by

$$J(1) = \frac{4\varepsilon_0}{9D^2}\left(\frac{2e}{m}\right)^{1/2} V^{3/2}, \tag{1}$$

where e and m are, respectively, the charge and mass of the emitted particle, and ε_0 is the free space permittivity. Implicit in Eq. (1) is the neglect of relativistic effects and the assumption of zero electron emission velocity. While Eq. (1) was published ninety years ago, no analogous derivation for two dimensions (2D) has appeared in the intervening years. This paper partially fills this void.

The complete solution of the 2D limiting current density in a diode is extremely difficult to obtain analytically. It requires the simultaneous solution of the force law, the continuity equation, and the Poisson equation in two dimensions. Such 2D theories are very complicated [4]; the results are neither transparent nor readily usable [5]. However, this 2D problem is of fundamental interest because electron emission is often restricted to a finite patch on the cathode surface. For example, modern cathodes, such as ferroelectric cathodes, laser-triggered cathodes, field emitter arrays, etc., have at times displayed an emission current density higher than that expected from the classical 1D Child-Langmuir value [Eq. (1)]. The proper interpretation of such results would have required, at a minimum, an understanding of the 2D limiting current density. This 2D problem is clearly also relevant to the edge emission in high power diodes.

In this paper, we shall not seek a complete 2D solution to the force law, continuity equation, and the Poisson equation. Instead, we simply derive the condition for the onset of turbulent behavior when a finite patch of the cathode surface is allowed to emit, with a uniform emission current density across the patch. The theory is simple and intuitive, and is in excellent agreement with earlier simulation results [6]. As we shall see toward the end of this paper, the physical insight even leads to the prediction of the limiting current density for a *three-dimensional* geometry.

A few years ago, the lack of a 2D Child-Langmuir solution prompted Luginsland *et al.* [6] to use two very different particle-in-cell codes to simulate the maximum current density, $J(2)$. The emission current density is uniform and is restricted to a strip of width W on the cathode surface. The strip is infinitely long. From the simulation data, the following 2D Child-Langmuir Law was synthesized [6]:

$$\frac{J(2)}{J(1)} \cong 1 + 0.3145\,\frac{D}{W}, \tag{2}$$

where D is the anode-cathode separation. Luginsland found that Eq. (2) fits the numerical data to within a few percent for all $W/D > 0.1$. This statement is valid regardless of the external magnetic field (ranging from zero to 100 T) that is imposed along the electron flow direction. When the emission current density exceeds the value given in Eq. (2), the emitted electrons from the center line of the emission strip are the first ones to be reflected, initiating a virtual cathode [6]. It is clear from Eq. (2) that the 1D Child-Langmuir law is recovered in the limit $W \gg D$.

We shall first provide a derivation of Eq. (2), under the assumption $W \gg D$. In the model, the cathode is located at $z = 0$, and the anode is located at $z = D$. Electrons are emitted with a uniform current density, J, over the infinite strip, $-W/2 < x < W/2$, on the cathode. We impose an infinite magnetic field in the z direction. We shall focus mainly on the space charge field on the center line, $(x, z) = (0, 0)$, of this emission strip since, as J is increased to the 2D limiting value, the space charge field there exactly cancels the vacuum electric field, V/D, causing reflection of the emitted electrons.

Let $\rho(x, z)$ be the charge density within the gap. An incremental line charge, located at (x, z) with a line charge density $\rho(x, z)\Delta x \Delta z$, yields an electric field at $(x, z) = (0, 0)$ with a magnitude

$$\Delta E = \frac{\rho(x, z)\Delta x \Delta z}{2\pi\varepsilon_0\sqrt{x^2 + z^2}}. \tag{3}$$

On the center line, we need only to consider the z component of the electric field by symmetry, and this component

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is obtained by multiplying Eq. (3) by the directional cosine, $z/(x^2 + z^2)^{1/2}$. Thus, the space charge field on the center line of the emitting strip is given by, upon summing over the space charge within the gap,

$$E = \int_0^D dz \int_{-W/2}^{W/2} dx \, \frac{\rho(x,z)z}{2\pi\varepsilon_0(x^2 + z^2)} . \tag{4}$$

We now assume that $W \gg D$, so that the charge density ρ is roughly independent of x. With $\rho(x,z) = \rho(z)$, the x integration in Eq. (4) can be performed to yield

$$\begin{aligned} E &= \frac{1}{\pi\varepsilon_0} \int_0^D dz\, \rho(z) \tan^{-1}\left(\frac{W}{2z}\right) \\ &\approx \frac{1}{2\varepsilon_0} \int_0^D dz\, \rho(z) \left(1 - \frac{4z}{\pi W}\right), \end{aligned} \tag{5}$$

where, in writing the last expression of Eq. (5), we have used the approximation, $\tan^{-1}(p) = \pi/2 - 1/p$ for $p \gg 1$. Since the injection current density, $J = \rho(z)v(z) =$ constant, where $v(z)$ is the electron velocity, we may write the total electric field, due only to the space charge, as

$$E = G\left(\frac{J}{2\varepsilon_0}\right) \int_0^D \frac{dz}{v(z)} \left(1 - \frac{4z}{\pi W}\right), \tag{6}$$

where we have included a multiplication factor, G, to account for the contributions due to the image charges. Since E is the space charge electric field at a *single* location, it must be proportional to J, and we take G to be the proportionality constant defined by Eq. (6). The 2D limiting current density, $J(2)$, is reached when this total self-electric field equals the vacuum field, V/D. This gives

$$G\left(\frac{J(2)}{2\varepsilon_0}\right) \int_0^D \frac{dz}{v(z)} \left(1 - \frac{4z}{\pi W}\right) = \frac{V}{D} . \tag{7}$$

The 1D Child-Langmuir law is obtained by setting W equal to infinity. Thus, by definition, we have, from Eq. (7),

$$G\left(\frac{J(1)}{2\varepsilon_0}\right) \int_0^D \frac{dz}{v(z)} = \frac{V}{D} . \tag{8}$$

Note that the value of G in Eq. (8) may easily be obtained by using Eq. (1) and the familiar 1D Child-Langmuir velocity solution $v(z)$ in Eq. (8). Note further that this value of G takes into account, exactly, the effects of all image charges in this 1D limit. We assume that this G is also applicable for the large but finite value of W [cf. Eq. (7)], and this latter assumption is on the same footing as approximating $\rho(x,z)$ by $\rho(z)$ in Eqs. (4) and (5).

We take the ratio of Eqs. (7) and (8) to obtain, to first order in $1/W$,

$$\frac{J(2)}{J(1)} \cong 1 + \frac{\int_0^D \frac{dz}{v(z)} \left(\frac{4z}{\pi W}\right)}{\int_0^D \frac{dz}{v(z)}} . \tag{9}$$

Since the last term in Eq. (9) is already a correction due to 2D effects, we may use the 1D Child-Langmuir velocity profile $v(z)$ in that term. From the well-known density profile, $\rho(z) \sim z^{-2/3}$, in the 1D Child-Langmuir solution, we obtain $v(z) = J/\rho(z) = Cz^{2/3}$ for some constant C. Using this 1D velocity profile in Eq. (9), we obtain

$$\frac{J(2)}{J(1)} \cong 1 + \frac{D}{\pi W} , \tag{10}$$

which is in remarkable agreement with the empirical formula [Eq. (2)].

While Eq. (10) is derived under the assumption $W/D \gg 1$, the simulation data [6] show that it happens to be applicable even if W/D is as small as 0.1.

It is therefore tempting to apply the same derivation to other geometries of emission. Before we do this, let us recapitulate the main assumptions used to derive Eq. (10): (a) $W/D \gg 1$; (b) $\rho(x,z) = \rho(z)$; (c) the image charge factor, G, in Eqs. (7) and (8), is the same whether W is finite or infinite; and (d) the 1D Child-Langmuir solution is used to approximate the correction due to 2D effects.

For the case where electron emission is restricted to a circular patch of radius R on the cathode, we may use exactly the same procedure to arrive at the following 2D Child-Langmuir law in the limit $R/D \gg 1$:

$$\frac{J(2)}{J(1)} \cong 1 + \frac{D}{4R} . \tag{11}$$

Note the similarity between Eqs. (10) and (11). From the excellent agreement between Eq. (10) and Eq. (2), it would not be surprising if Eq. (11) happens to give an excellent approximation for $R/D < 1$. Indeed, Luginsland [7] recently confirmed that Eq. (11) agrees with simulation data to within a few percent whenever $R/D > 0.5$.

The simple scaling laws, Eqs. (10) and (11), allow us to postulate the maximum current density, $J(3)$, when the emitting area on the planar cathode is an ellipse with semiaxes R and $W/2$, with the restriction $R > W/2$. This ellipse reduces to a circle of radius R in the $R = W/2$ limit, and to an infinite strip of width W in the $R \gg W$ limit. Thus, we postulate

$$\begin{aligned} \frac{J(3)}{J(1)} &\cong 1 + \frac{D}{\pi W} + \left(\frac{1}{4} - \frac{1}{2\pi}\right) \frac{D}{R} \\ &= 1 + 0.32 \frac{D}{W} + 0.091 \frac{D}{R} , \qquad R \geq \frac{W}{2} , \end{aligned} \tag{12}$$

so that Eq. (12) reduces to Eq. (10) in the $R \gg W$ limit, and to Eq. (11) in the $R = W/2$ limit. In Eq. (12), we use $J(3)$ to denote the limiting current density because to simulate emission from an elliptical patch into a planar gap would have required a three-dimensional particle-in-cell code.

The scaling laws derived thus far predict the emission current density for the onset of the virtual cathode, which occurs at the center region of the emission area. If the available electrons are further increased (e.g., by raising the laser intensity in a photocathode [8]), a point may be reached where the entire emitting area may become space-charge-limited and the electron emission becomes saturated. This latter situation was recently analyzed in Refs. [5,9], and it was found that the current density peaks at the rim of the emitting region (i.e., the current density profile has a "winglike" structure). The most recent photocathode experiments [8] seem to have confirmed both the onset of virtual cathode and the saturated emission as the laser intensity is raised, in a manner described in this paragraph.

While Eqs. (10) and (11) agree well with simulation data down to relatively small values of W and R, it must be stressed that they do *not* give the correct limits when W and R approach zero. One can prove that, as W approaches zero, $J(2)/J(1) < A(D/W)^a$ for some constants A and a with $a < 1$. One can similarly prove that, as R approaches zero, $J(2)/J(1) < B(D/R)^b$ for some constants B and b with $b < 2$. It would be highly desirable to establish the asymptotic limits of $J(2)/J(1)$ as W and R approach zero.

In conclusion, a simple 2D Child-Langmuir law is established analytically for the first time. It is in excellent agreement with numerical simulations. It predicts the onset of virtual cathode before the entire emitting region reaches saturation. The simple scaling laws are established from first principles. They are independent of the cathode materials, as in the 1D case.

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