Manifestation of a Nontrivial Vacuum in Discrete Light-Cone Quantization

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We study a $(1 + 1)$-dimensional $\lambda\phi^4$ model with a light-cone zero mode and constant external source to describe spontaneous symmetry breaking. In the broken phase, we find degenerate vacua and discuss their stability based on effective-potential analysis. The vacuum triviality is spurious in the broken phase because these states have lower energy than Fock vacuum. Our results are based on the variational principle.

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Light-cone quantization [1] has been studied to clarify nonperturbative aspects of field theories [2] and used to provide nonperturbative formulation of M theory [3]. This framework simplifies dynamics of quantum field theories since it prohibits vacuum diagrams kinematically [4]. It has also a possibility of calculating wave functions of physical states in a nonperturbative manner. Because of these preferable properties, we hope for this framework to play a complementary role in lattice theories.

DLCQ (discrete light-cone quantization) is the most well-defined treatment of light-cone quantization, which enables a clear separation of the zero mode [5–7]. It has been believed that the true vacuum is trivial and only the zero mode is responsible for spontaneous symmetry breaking (SSB) in scalar field theories [5,8]. There are some studies that rely on a combination of constrained zero modes and trivial vacuum [9]. The vacuum triviality, however, results from an assumption that normal-ordered Hamiltonians are positive semidefinite. In this Letter, we show that this assumption is not always true. We include a zero mode and external source in analytic variational calculations to show the existence of nontrivial vacuum with lower energy than trivial Fock vacuum. Our results are based on previous works on zero-mode singularity [8] and quantum solitons [10,11] in DLCQ. We examine a possibility that zero modes reproduce SSB, which has not been considered in Refs. [10,11].

In order to define an effective potential, we consider the generating functional of Green's functions, $W[J] = -i\ln Z[J]$. We can express it using a Hamiltonian $H[J]$ when the external source is time independent $J(x) = J(\boldsymbol{x})$ [10,12],

$$H[J]|0_J\rangle = w[J]|0_J\rangle,$$
$$H[J] = H - \int_{-L}^{L} d\boldsymbol{x}\, J(\boldsymbol{x})\phi(\boldsymbol{x}),$$

where $w[J] = -W[J]/T$ is energy of the ground state $|0_J\rangle$. T is the time difference between the initial and final states. If we are interested just in the ground state (vacuum), we can consider an effective potential to reduce the problem into a simpler one. It is defined as a Legendre transform $V(\varphi_0) \equiv w(J) + J\varphi_0 = \langle 0_J|H|0_J\rangle/2L$ with a constant source $J(x) = J$ and vacuum-energy density $w(J) = w[J]/2L$. A vacuum expectation value (VEV) of a zero mode is given by $\varphi_0 = -dw(J)/dJ = \langle 0_J|\phi_0|0_J\rangle$. We can calculate the effective potential $V(\varphi_0)$ if the energy and the wave function of the ground state are known. Our purpose is to identify the true vacuum in the $(1 + 1)$-dimensional $\lambda\phi^4$ model with a double-well classical potential. We perform variational calculations for the following normal-ordered Hamiltonian:

$$H(J) = \int_{-L}^{L} dx^-\left(-\frac{\mu^2}{2} : \phi^2 : + \frac{\lambda}{4!} : \phi^4 : - J : \phi :\right). \tag{1}$$

Hereafter, we designate the space coordinate x^- by x. The field operator is decomposed into zero and nonzero modes $\phi(x) = \phi_0 + \tilde{\phi}(x)$, where $(O)_0 \equiv \int_{-L}^{L} dx\, O(x)/2L$. In DLCQ with periodic boundary conditions, the zero mode ϕ_0 is constrained [5],

$$\Phi[\tilde{\phi}] = -\mu^2 : \phi_0 : + \frac{\lambda}{6} : (\phi^3)_0 : - J = 0. \tag{2}$$

Operator ordering has been chosen so that it satisfies $dw(J)/dJ = -\varphi_0$. It is the consistency condition to be satisfied between the Hamiltonian (1) and the zero-mode constraint (2). We discuss it later in Eq. (24). The zero mode is an operator functional of the nonzero mode

$$\tilde{\phi}(x) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{4\pi n}}(a_n e^{-ip_n^+ x} + a_n^\dagger e^{ip_n^+ x}), \tag{3}$$

where $[a_m, a_n^\dagger] = \delta_{m,n}$ and $a_m|0_{\rm F}\rangle = 0$. We truncate the system by introducing the one-coherent state $|\tilde{\varphi}\rangle = e^{\sum_{n=1}^{\infty}(\tilde{\varphi}_n a_n^\dagger - \frac{1}{2}\tilde{\varphi}_n^*\tilde{\varphi}_n)}|0_{\rm F}\rangle$. It provides useful formulas $a_n|\tilde{\varphi}\rangle = \tilde{\varphi}_n|\tilde{\varphi}\rangle$ and $\langle\tilde{\varphi}| : O[\tilde{\phi}] : |\tilde{\varphi}\rangle = O[\tilde{\varphi}]$, where $\varphi(x) \equiv \langle\tilde{\varphi}|\phi(x)|\tilde{\varphi}\rangle = \varphi_0 + \tilde{\varphi}(x)$. Our variational parameter is $\tilde{\varphi}(x)$.

We take the continuum limit with $P^+ = \pi M/L$ fixed, where M is called the harmonic resolution. We include this constraint on P^+ in the Hamiltonian using the Lagrange's undetermined multiplier.

$$
\begin{aligned}
E &\equiv \langle\tilde{\varphi}|H_\beta(J)|\tilde{\varphi}\rangle \\
&= \int_{-L}^{L} dx \left\{ \beta\left[\left(\frac{d\tilde{\varphi}}{dx}\right)^2 - \frac{\pi M}{2L^2}\right] \right. \\
&\qquad \left. - \frac{\mu^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 - J\varphi \right\},
\end{aligned} \tag{4}
$$

where $w(J) = E/2L$. Equation (2) and the stationary condition for (4) give the following coupled equations for the zero and nonzero modes:

$$-\mu^2\varphi_0 + \frac{\lambda}{6}\varphi_0^3 = J, \tag{5}$$

$$-2\beta\frac{d^2\tilde{\varphi}}{dx^2} + \mu_e^2\tilde{\varphi} + \frac{\lambda}{2}\varphi_0\tilde{\varphi}^2 + \frac{\lambda}{6}\tilde{\varphi}^3 = 0, \tag{6}$$

where $\mu_e^2 \equiv -\mu^2 + \lambda\varphi_0^2/2$. We have the following zero-mode solutions to (5) when $J = 0$ (see Fig. 1):

$$\varphi_0 = 0, \pm v, \qquad v \equiv \sqrt{\frac{6\mu^2}{\lambda}}. \tag{7}$$

The system defined in a finite box can describe SSB if the vanishing J limit is taken after all calculations. We should choose $\varphi_0 = \pm v$ but not $\varphi_0 = 0$ as physical solutions to (5) when $J = 0$, because the solution $\varphi_0 = 0$ is not connected continuously to expectation values for large $J \neq 0$. The region $|\varphi_0| < v/\sqrt{3}$ is unphysical.

If one attempts to see vacuum physics without introducing an external source, wrong solutions may be obtained. The symmetric phase would be safe without an external source, but the broken phase is not. We explain how the true vacuum solution appears by paying attention to the effects of the zero mode and the external source on the ground state. We also discuss vacuum stability in Eq. (25) based on the effective potential.

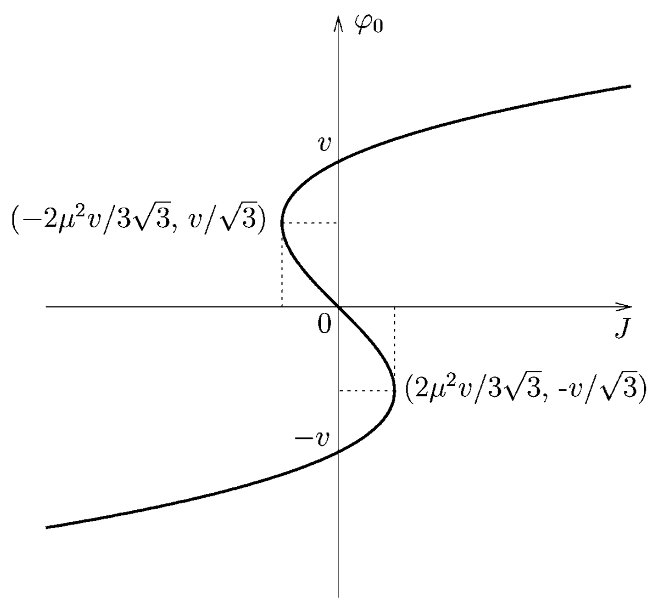

FIG. 1. Functional relationship (5) between φ_0 and J is shown when Fock space is truncated with the one-coherent state approximation. The region $|\varphi_0| < v/\sqrt{3}$ is unstable [see discussions given below Eq. (25)]. When $|\varphi_0| \geq v$, the zero mode φ_0 increases monotonously, $d\varphi_0/dJ > 0$.

Solutions to Eq. (6) must satisfy the following two conditions simultaneously: (i) The solution $\tilde{\varphi}$ to (6) and the nth derivative $\tilde{\varphi}^{(n)} = d^n\tilde{\varphi}/dx^n$ must be periodic at the boundaries $x = \pm L$: $\tilde{\varphi}^{(n)}(-L) = \tilde{\varphi}^{(n)}(L)$, $n = 0, 1, 2, \ldots.$ (ii) The solution $\tilde{\varphi}$ to (6) must be the nonzero mode: $[\tilde{\varphi}(x)]_0 = \int_{-L}^{L} dx\, \tilde{\varphi}(x)/2L = 0$.

For convenience, let us regard x as time and consider the following dummy Hamiltonian $\mathcal{H}$ that reproduces the equation of motion (6):

$$\mathcal{H} = \tfrac{1}{2}(\partial\tilde{\varphi})^2 + \mathcal{V}(\tilde{\varphi}), \tag{8}$$

where $\mathcal{V}$ is a dummy potential

$$
\begin{aligned}
\mathcal{V}(\tilde{\varphi}) = -\frac{1}{2\beta}\bigg(&\frac{\mu_e^2}{2}\tilde{\varphi}^2 + \frac{\lambda}{6}\varphi_0\tilde{\varphi}^3 + \frac{\lambda}{4!}\tilde{\varphi}^4 \\
&- \frac{\mu^2}{2}\varphi_0^2 + \frac{\lambda}{4!}\varphi_0^4\bigg).
\end{aligned} \tag{9}
$$

We first solve (6) especially when $J = 0$ and $\varphi_0 = 0$. As mentioned before, this gives physically unacceptable solutions. However, these solutions are technically helpful for the purpose of calculating the energy and the wave function of the true degenerate vacua with nonzero VEVs as shown later. When $J = 0$ and $\varphi_0 = 0$, Eq. (6) reduces to

$$\frac{d^2\tilde{\varphi}}{dx^2} = \frac{1}{2\beta}\left(-\mu^2\tilde{\varphi} + \frac{\lambda}{6}\tilde{\varphi}^3\right), \tag{10}$$

and the dummy potential $\mathcal{V}$ is

$$\mathcal{V}(\tilde{\varphi}) = -\frac{1}{2\beta}\left(-\frac{\mu^2}{2}\tilde{\varphi}^2 + \frac{\lambda}{4!}\tilde{\varphi}^4\right). \tag{11}$$

When the parameter β is positive, the dummy potential $\mathcal{V}(\tilde{\varphi})$ is bounded from above [see Fig. 2(a)]. From

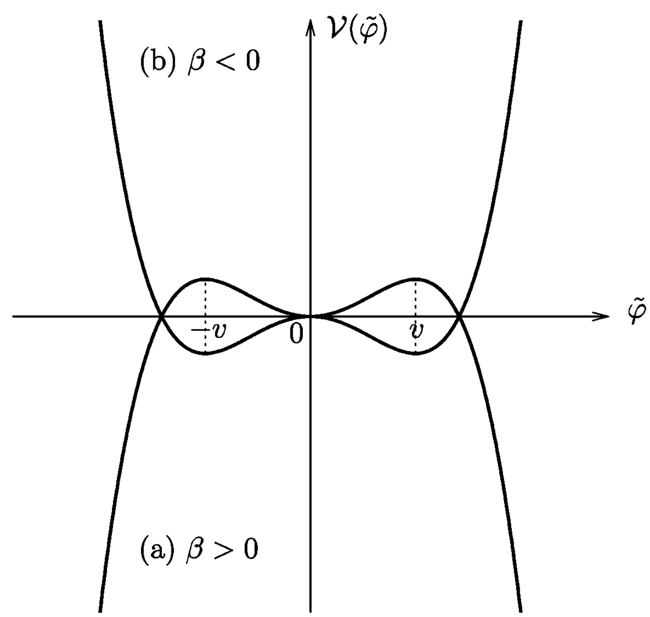

FIG. 2. The dummy potential $\mathcal{V}$ is drawn as a function of $\tilde{\varphi}$ when $\varphi_0 = 0$: (a) positive β and (b) negative β.

condition (i), the motion of a particle must be periodic. Namely, the particle must reside between the two maximums of the dummy potential. If condition (i) is satisfied, condition (ii) is also satisfied since the particle oscillates around the origin in the symmetric dummy potential. The solution to (10) is

$$\tilde{\varphi}_{\rm sn}(x) = \left[\frac{12\mu^2 k^2}{\lambda(k^2+1)}\right]^{1/2} \mathrm{sn}(a_{\rm sn}x, k), \qquad (12)$$

where sn is a Jacobian elliptic function and $0 \le k \le 1$ [11]. The values of the parameters $a_{\rm sn}$ and k are determined so that the solution (12) satisfies both conditions (i) and (ii); $k = 0$ gives $\tilde{\varphi}(x) = 0$ and $M = 0$, which correspond to trivial Fock vacuum with $\varphi_0 = 0$ and $P^- = 0$. $k = 1$ is not acceptable since it gives $\tilde{\varphi} \sim \tanh(ax)$, which is an odd function and cannot satisfy the periodicity condition (i). When $0 < k < 1$, we have $a_{\rm sn} = 2NK(k)/L$ [N is a natural number and $K(k)$ is the complete elliptic integral of the first kind] from the periodicity condition (i) and it satisfies condition (ii). In this case, the solution (12) is acceptable and gives the following energy and harmonic resolution:

$$\frac{E_{\rm sn}}{2L} = -\frac{6\mu^4 k^2}{\lambda(k^2+1)^2} + \frac{576N^2\mu^6 k^4 I_{\rm sn}^2(k)}{\pi\lambda^2(k^2+1)^3 M}, \qquad (13)$$

$$M = \frac{96N\mu^2 k^2 I_{\rm sn}(k)K(k)}{\pi\lambda(k^2+1)}, \qquad (14)$$

$$I_{\rm sn}(k) \equiv \int_0^1 df\sqrt{(1-f^2)(1-k^2f^2)}. \qquad (15)$$

In the limit $k \to 1$ with $N = 1$, the energy (13) takes the minimum value, and the harmonic resolution goes to infinity since $K(k)$ diverges at $k = 1$ giving the continuum limit $M \to \infty$. This is the solution given in Ref. [11].

When the parameter β is negative, condition (i) is automatically satisfied since the dummy potential $\mathcal{V}(\tilde{\varphi})$ is bounded from below and a particle oscillates necessarily with a fixed period [see Fig. 2(b)]. Equation (10) has two types of solutions in this case. We can also express them using Jacobian elliptic functions $\mathrm{cn}(x,k)$ and $\mathrm{dn}(x,k)$. When a particle oscillates around the origin, a solution is

$$\tilde{\varphi}_{\rm cn}(x) = \left[\frac{12\mu^2 k^2}{\lambda(2k^2-1)}\right]^{1/2} \mathrm{cn}(a_{\rm cn}x, k). \qquad (16)$$

The solution (16) is acceptable as an exited state since it satisfies conditions (i) and (ii) simultaneously with higher energy than (12). When a particle oscillates around one of the minimums of the dummy potential, solutions are

$$\tilde{\varphi}_{\rm dn}(x) = \pm\left[\frac{12\mu^2}{\lambda(2-k^2)}\right]^{1/2} \mathrm{dn}(a_{\rm dn}x, k). \qquad (17)$$

The solutions (17) are not acceptable since they cannot satisfy condition (ii).

When $J = 0$ and $\varphi_0 = 0$, the candidate solution for the ground state is (12) with $a_{\rm sn} = 2NK(k)/L$ and $k \to 1$ derived for positive β since it gives the lowest energy. However, we discard the solution (12) since the state formed by it cannot be connected to solutions for the nonzero external source $J \neq 0$. In addition, mass squared goes to negative infinity $P^\mu P_\mu \to -\infty$ in the continuum limit $M \to \infty$ since the first term of the energy (13) is nonzero and negative. This is the reason why all past calculations have not been successful in calculating mass squared stably in the broken phase [10].

When the limit $J \to 0$ is taken starting from a sufficiently large J, an effective potential chooses one of $\varphi_0 = \pm v$ depending on the sign of J. Nonzero values of J resolve the degeneracy of the two vacua, which is restored in the limit $J \to 0$. In this case, conditions (i) and (ii) select dn-type oscillation around the minimum $\tilde{\varphi} = 0$ of the dummy potential. The situation is completely different from the $\varphi_0 = 0$ case. We obtain the following dummy potential by substituting $\varphi_0 = v$ into (9) (it is enough to consider one of the two degenerate vacua $\varphi_0 = \pm v$):

$$\mathcal{V}(\tilde{\varphi}) = -\frac{1}{2\beta}\left(\mu^2\tilde{\varphi}^2 + \frac{\lambda}{6}v\tilde{\varphi}^3 + \frac{\lambda}{4!}\tilde{\varphi}^4 - \frac{3\mu^4}{2\lambda}\right). \qquad (18)$$

This is the case when the nonzero mode is shifted with $\tilde{\varphi} \to \tilde{\varphi} + v$ in (11). Therefore, we obtain four possible oscillations by shifting the solutions (12), (16), and (17) with $\tilde{\varphi} \to \tilde{\varphi} - v$. The first is an sn-type oscillation around $\tilde{\varphi} = -v$. The second is a cn-type oscillation around $\tilde{\varphi} = -v$. The third is a dn-type oscillation around $\tilde{\varphi} = -2v$. However, all of them are unacceptable since they cannot satisfy condition (ii). The following dn-type solution is a physically acceptable oscillation:

$$\tilde{\varphi}(x) = \left[\frac{12\mu^2}{\lambda(2-k^2)}\right]^{1/2} \mathrm{dn}(ax, k) - v, \qquad (19)$$

which oscillates around the origin $\tilde{\varphi} = 0$ and can satisfy conditions (i) and (ii) simultaneously. Its energy and total momentum are

$$\frac{E}{2L} = \frac{6\mu^4(k^2-1)}{\lambda(2-k^2)^2} - \frac{144\mu^6 I_{\rm dn}^2(k)}{\pi\lambda^2(2-k^2)^3 M}, \qquad (20)$$

$$P^+ = \frac{\pi M}{L} = \frac{24\mu^2 I_{\rm dn}(k)}{\lambda(2-k^2)}\, a, \qquad (21)$$

$$I_{\rm dn}(k) \equiv \int_{\mathrm{dn}(aL,k)}^1 df\sqrt{(1-f^2)(f^2-1+k^2)},$$

where $0 \le k \le 1$; $k = 0$ is not acceptable since $\tilde{\varphi}(x) = 0$ gives $\varphi_0 = \langle 0_{\rm F}| : \phi_0 : |0_{\rm F}\rangle = 0$ that contradicts $\varphi_0 = v$. When $0 < k < 1$, there exists no value of the parameter $a = NK(k)/L$ that can satisfy condition (ii). When $k = 1$, we have

$$\tilde{\varphi}(x) = v\left[\frac{\sqrt{2}}{\cosh(ax)} - 1\right]. \qquad (22)$$

Condition (ii) requires the following relation to hold:

$$\sqrt{2}\,\mathrm{gd}(aL) - aL = 0, \qquad (23)$$

where gd is the Gudermann function. This has a solution $aL \sim 1.72$. In the continuum limit $L \to \infty$, we have $a \to$

0 which gives zero total momentum $P^+ = 0$. Since the energy (20) is negative and lower than Fock vacuum in the continuum limit, we identify it as one of the true vacua.

In order to check the consistency of the operator ordering between the Hamiltonian (1) and the zero-mode constraint (2), we examine the first derivative of $w(J)$ with respect to J using (5) and (6).

$$\frac{dw}{dJ} + \varphi_0 = \int_{-L}^{L} \frac{dx}{2L} \left[\frac{d\tilde{\varphi}}{dJ} \left(\mu_e^2 \tilde{\varphi} + \frac{\lambda}{2} \varphi_0 \tilde{\varphi}^2 + \frac{\lambda}{6} \tilde{\varphi}^3 \right) + \frac{d\varphi_0}{dJ} \Phi[\tilde{\varphi}] \right] = -\beta \frac{d}{dJ} \left(\frac{P^+}{2L} \right). \tag{24}$$

Since the parameters P^+ and L are given by hand independent of J, we obtain the desired consistency condition $dw/dJ = -\varphi_0$. We can use this relation to discuss vacuum stability also. We have the following relation from the definition of the effective potential $V(\varphi_0) = w(J) + J\varphi_0$:

$$\frac{d^2V}{d\varphi_0^2} = \frac{dJ}{d\varphi_0}. \tag{25}$$

The state given by (22) is stable and hence can be regarded as one of the true vacua since (25) is positive when $\varphi_0 = v$. On the other hand, the state given by (12) is unstable since (25) is negative when $\varphi_0 = 0$. In $-v/\sqrt{3} < \varphi_0 < v/\sqrt{3}$, energy decreases as J increases. If ϕ_0 or J is not introduced, one cannot observe this instability of the state (12) since $V(\varphi_0)$ and hence (25) are not available.

We conclude that there exist nontrivial degenerate vacua other than Fock vacuum in the $(1 + 1)$-dimensional $\lambda\phi^4$ model with a double-well classical potential. We have shown stability of the obtained vacua based on the effective-potential analysis. The essential point of our analysis is introduction of a zero mode and an external source.

In general, there are singlet and nonsinglet sectors of Z_2 symmetry. One-coherent state $|\tilde{\varphi}\rangle$ is a mixed state of both the sectors. We have shown that there exist nonsinglet vacua with lower energy than Fock vacuum when the classical potential has a double-well shape. The mixing of the singlet and nonsinglet sectors is a consequence of the introduction of an explicitly symmetry-breaking interaction $J\phi_0$ in the Hamiltonian (1).

The issues of critical exponents remain still open until quantitatively reliable calculations are done. We should perform variational calculations without assuming vacuum triviality also in the case when a classical potential is convex. However, we need to interpret the difference among normal (ours) and other operator orderings (such as Weyl ordering) concerning the origin of VEVs of zero modes. When the Hamiltonian and zero-mode constraint are normal ordered, trivial Fock vacuum gives just $\langle 0_F| : H : |0_F\rangle = 0$ and $\langle 0_F| : \phi_0 : |0_F\rangle = 0$ (i.e., there is no SSB if vacuum triviality is assumed for normal-ordered operators).

Finally, we point out the importance of small momentum components near the zero mode. The reason why the true-vacuum solution (22) gives $P^+ = 0$ in the continuum limit $L \to \infty$ is that its slowly changing configuration is mainly composed of small momenta. This is an extended description of the accumulating point discussed before in Ref. [13]. On the other hand, the solution (12) with $a_{\mathrm{sn}} = 2NK(k)/L$ and $k \to 1$ needs large momentum components to describe its singular behavior at $x = 0$ and $\pm L$, which gives infinite harmonic resolution $M \to \infty$.

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