Teleportation as a Depolarizing Quantum Channel, Relative Entropy, and Classical Capacity

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We show that standard teleportation with an arbitrary mixed state resource is equivalent to a generalized depolarizing channel with probabilities given by the maximally entangled components of the resource. This enables the usage of any quantum channel as a generalized depolarizing channel without additional twirling operations. It also provides a nontrivial upper bound on the entanglement of a class of mixed states. Our result allows a consistent and statistically motivated quantification of teleportation success in terms of the relative entropy and this quantification can be related to a classical capacity.

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The possibility of transferring an unknown quantum state using preexisting entanglement and a classical information channel was labeled *teleportation* by its authors [1]. The teleportation process can be viewed as a quantum channel. The nature of the channel is determined by both the state used as a teleportation resource and the particular protocol used with this resource [2–4]. The standard teleportation protocol T_0 using the Bell diagonal measurements and Pauli rotations, when used in conjunction with a Bell state resource, provides an example of a *noiseless* quantum channel $\Lambda_{T_0}(|\Psi^+\rangle\langle\Psi^+|)\varrho = \varrho$. Teleportation using mixed states as an entanglement resource is, in general, equivalent to a *noisy* quantum channel. A general expression for the *output state* of a teleportation process with an *arbitrary* mixed resource, in terms of some quantum channel, has been shown previously [4]. In this Letter, we derive an explicit expression for the quantum channel associated with the standard teleportation protocol on a mixed state resource. Our result establishes a many to one correspondence between arbitrary bipartite quantum states and generalized depolarizing channels. This is a complete generalization of an earlier correspondence noted by the Horodecki's [2] between quantum channels Λ and the *restricted class* of quantum states ρ_Λ with one reduced density matrix equal to the maximally mixed state. We then present both practical and theoretical applications of our result. From a practical point of view, our result allows an arbitrary quantum channel to be used as a generalized depolarizing channel. It permits Bell diagonal states to be shared between ends of an arbitrary channel without resorting to the time-consuming twirling operations [2,5]. On the theoretical side, our result can be used to obtain a *nontrivial* upper bound on the entanglement of a certain class of mixed states. We then show that our result allows the quantification of the success of teleportation consistently (i.e., without any divergence) in terms of the relative entropy. This quantification *unifies* the methodology of quantification of teleportation success with that of entanglement [6,7] and classical capacity [8–12]. More importantly, it gives a statistical interpretation of the teleportation success, with *collective measurements* being allowed on an ensemble of N teleported states after N separate teleportation processes [6]. The currently used *fidelity* of teleportation [2] fails to do this. In the end we show that teleportation success, as quantified by the relative entropy, is bounded above by a classical capacity. This relation can be regarded as connecting a quantum and a classical capacity.

We start by stating the main result of the Letter before going into its proof. It states that the standard teleportation protocol T_0, when used with an arbitrary two qubit mixed state, χ, as a resource, acts as a generalized depolarizing channel,

$$\Lambda_{T_0}(\chi)\varrho = \sum_i \mathrm{Tr}[E^i\chi]\sigma^i\varrho\sigma^i, \tag{1}$$

where the E^i's are the Bell states associated with the Pauli matrices σ^i, by $E^i = \sigma^i E^0 \sigma^i$, where $E^0 = |\Psi^+\rangle\langle\Psi^+|$ and $\sigma^0 = I, \sigma^1 = \sigma_x, \sigma^2 = \sigma_y$, and $\sigma^3 = \sigma_z$.

The result generalizes the relationship between particular teleportation protocols and quantum channels to include *all* 2×2 mixed states and proves the conjecture (made in Ref. [2]) that the relationship between mixed states used for teleportation and the resultant quantum channel is not one to one. The derivation, it may be noted, rests critically on the linearity of the teleportation protocol [1]. We also extend the result to teleportation with $d \times d$ state systems. We next proceed to the derivation of our central result.

Suppose Alice wishes to teleport the unknown qubit ϱ, then initially we can extend this state to a 2×2 pure state $|\psi\rangle_{12}$, even if ϱ is initially pure, such that $\mathrm{Tr}_2[|\psi\rangle\langle\psi|_{12}] = \varrho$. We then teleport only the original state ϱ and examine the outcome by comparing the total state $|\psi\rangle$ to the entanglement swapped state. Since an arbitrary state in a 2×2 system may be written in terms of a superposition of Bell basis states,

$$|\psi\rangle = c_0|\Psi^+\rangle + c_1|\Psi^-\rangle + c_2|\Phi^+\rangle + c_3|\Phi^-\rangle, \tag{2}$$

and because of the linearity of the teleportation protocol, we need only look at how the component Bell states of the density matrix $|\psi\rangle\langle\psi|$ are affected by teleportation using the T_0 protocol, using an arbitrary resource ρ.

We label the 2×2 state used in the teleportation by $|\psi\rangle_{12}$ and the resource by χ_{34}, where the subscripts denote the particle number. Alice has qubits in states $\chi_3 = \mathrm{Tr}_4[\chi_{34}]$ and $\varrho_1 = \mathrm{Tr}_2[|\psi\rangle\langle\psi|_{12}]$, and Bob has a qubit in the state $\chi_4 = \mathrm{Tr}_3[\chi_{34}]$. The outcome of the teleportation is the state

$$\Lambda_{T_0}(\chi_{34})\,|\psi\rangle\langle\psi|_{12} = \omega_{24}\,. \tag{3}$$

Choosing the basis state $|\psi\rangle_{12} = |\Psi^+\rangle$, in Eq. (3), we note that the teleportation then becomes a version of entanglement swapping [1,13,14] with one perfect and one noisy entangled state. Given a measurement outcome of the ith state upon measurement, we know that the final state, before the unitary operation, is in the state $\omega_{24}^i = \sigma_2^i \chi_{24} \sigma_2^i$, because this is equivalent to teleportation with the state $|\Psi^+\rangle_{12}$, without applying the unitary transform σ_2^i to the output state, and $\sigma^i = (\sigma^i)^\dagger = (\sigma^i)^{-1}$.

As the teleportation uses the channel χ_{34}, the unitary operation is applied to χ_4, and the output state is then

$$\omega_{24}^i = \sigma_4^i \sigma_2^i \chi_{24} \sigma_2^i \sigma_4^i\,, \tag{4}$$

and therefore, over all outcomes i, the final total teleported state is

$$\omega_{24} = \sum_i p_i \omega_{24}^i = \sum_i p_i \sigma_4^i \sigma_2^i \chi_{24} \sigma_2^i \sigma_4^i\,, \tag{5}$$

where p_i is the chance of obtaining outcome i upon measurement.

A tedious calculation shows that the probability of gaining outcome i, for the combined Bell state measurements on qubits 1 and 3, is simply $p_i = 1/4$. Hence, we can move the summation to obtain

$$\omega_{24} = \frac{1}{4}\sum_i \sigma_4^i \sigma_2^i \chi_{24} \sigma_2^i \sigma_4^i \tag{6}$$

$$= \sum_i E_{24}^i \chi_{24} E_{24}^i \tag{7}$$

$$= \sum_i \mathrm{Tr}[E_{34}^i \chi_{34}] E_{24}^i \tag{8}$$

$$= \sum_i \mathrm{Tr}[E_{34}^i \chi_{34}] \sigma_4^i |\Psi^+\rangle\langle\Psi^+|_{24} \sigma_4^i\,. \tag{9}$$

The equality between Eq. (6) and Eq. (7) can be shown by decomposing the Pauli operators in Eq. (6) in terms of the Bell state projectors, for example, $\sigma_2^1 \sigma_4^1 = E_{24}^0 + E_{24}^1 - E_{24}^2 - E_{24}^3$, and noting that all terms except those of the form given in Eq. (7) cancel.

Substituting another Bell state E_{12}^j into Eq. (3) simply rotates the output state by the corresponding Pauli operator $\sigma_2^j \omega_{24}^i \sigma_2^j$, and so

$$\omega_{24}^{(j)} = \sigma_2^j \left(\sum_i \mathrm{Tr}[E_{34}^i \chi_{34}] E_{24}^i \right) \sigma_2^j \tag{10}$$

$$= \sum_i \mathrm{Tr}[E_{34}^i \chi_{34}] \sigma_4^i E_{24}^j \sigma_4^i\,. \tag{11}$$

Additionally, the off diagonal Bell terms, $F^{mn} = \sigma^m |\Psi^+\rangle\langle\Psi^+| \sigma^n$, for $m \neq n$, follow by the linearity of the standard teleportation protocol,

$$\omega_{24}^{(mn)} = \sigma_2^m \left(\sum_i \mathrm{Tr}[E_{34}^i \chi_{34}] E_{24}^i \right) \sigma_2^n \tag{12}$$

$$= \sum_i \mathrm{Tr}[E_{34}^i \chi_{34}] \sigma_4^i F_{24}^{mn} \sigma_4^i\,. \tag{13}$$

The total final teleported state, given an arbitrary state $|\psi\rangle\langle\psi|_{12}$ as input, is then

$$\begin{aligned}\omega_{24} = {} & |c_0|^2 \sum_i \mathrm{Tr}[E_{34}^i \chi_{34}] \sigma_4^i |\Psi^+\rangle\langle\Psi^+|_{24} \sigma_4^i \\ & + \sum_{j\neq 0} |c_j|^2 \sum_i \mathrm{Tr}[E_{34}^i \chi_{34}] \sigma_4^i E_{24}^j \sigma_4^i \\ & + \sum_{m\neq n} c_m c_n^* \sum_i \mathrm{Tr}[E_{34}^i \chi_{34}] \sigma_4^i F_{24}^{mn} \sigma_4^i \end{aligned} \tag{14}$$

$$= \sum_i \mathrm{Tr}[E_{34}^i \chi_{34}] \sigma_4^i |\psi\rangle\langle\psi|_{24} \sigma_4^i\,, \tag{15}$$

and by tracing over qubit 2 in Eq. (15) and comparing with Eq. (1) we can see that the channel acts as a generalized depolarization channel,

$$\Lambda_{T_0}(\chi)\varrho = \sum_i p_i \sigma^i \varrho \sigma^i\,, \tag{16}$$

with the probabilities given by the projections of the Bell states on the teleportation resource $p_i = \mathrm{Tr}[E^i \chi]$. The above result has been proved so far only for the teleportation of state ρ of a single qubit. From Eq. (15) and *linearity,* it can easily be extended to the case of teleportation of one-half of a 2×2 mixed state γ_{12} (i.e., for entanglement swapping) through a bipartite resource χ_{34}. We simply have to replace $|\psi\rangle\langle\psi|_{24}$ in Eq. (15) by γ_{24} in order to obtain the output state of the teleportation process. Teleportation of an entangled mixed state γ_{12} is thus given by

$$\Lambda_{T_0}(\chi)\gamma = \sum_i \mathrm{Tr}[E^i \chi] \sigma_4^i \gamma_{24} \sigma_4^i\,. \tag{17}$$

Equations (16) and (17) are the first ever general expressions for teleportation and entanglement swapping with *arbitrary* mixed states, as long as the teleportation protocol is kept standard. One must remember that for optimal utilization of a given entangled resource, one must choose local basis states such that p_0 is maximum. One can regard this particular state as the principal state ($|\Psi^+\rangle$) of the teleportation protocol. In principle, it should allow one to rederive all known results about the standard protocol (for example, the dependence of teleportation fidelity on the maximally entangled fraction [2] only). However, in the rest of this Letter, we explore those consequences of our result which are unknown to date.

Equation (17) immediately provides an upper bound to the entanglement of a class of mixed states. From the fact that entanglement cannot be increased under local actions and classical communications, it follows that the output entangled state $\lambda = \Lambda_{T_0}(\chi)\gamma$ must have an entanglement lower than that of the less entangled of the states γ_{12} and χ_{34}. Therefore, for any state λ_{12} expressible in terms of another state γ_{12} as $\sum_i p_i \sigma_2^i \gamma_{12} \sigma_2^i$, the entanglement

$$\mathcal{E}(\lambda) \leq \mathcal{E}(\beta\{p_i\}), \tag{18}$$

where $\beta\{p_i\}$ denotes the class of states with Bell diagonal projections p_i. This bound implies that for generlized qubit depolarizing channels, Λ, with a spectrum $p_i \in [0, 1/2]$, we have $\Lambda\rho$ to be separable for all ρ. In other words, no matter what initial state you use, you can *never* establish entanglement between the ends of such a channel. When $\beta\{p_i\}$ are taken to be Bell diagonal states, the *upper* bound of Eq. (18) will *complement* the usual *lower* bounds on entanglement of states obtained by wernerization [5]. The above bound implies that the entanglement left after passing one-half of an arbitrary mixed entangled state γ through a generalized depolarizing channel is less than or equal to that left when one-half of a Bell state is passed through the channel. The *nontriviality* of the result stems from the fact that even if γ is obtainable from a Bell state by action of local operators, these operators do not necessarily commute with those of the depolarizing channel.

The next noteworthy consequence of our result is that it provides an alternative to the use of time-consuming twirling operations [2,5] in quantum communication protocols. Such operations involve applying random local unitary operations to an entangled pair of particles to bring them to a Bell diagonal state. Here, first, there is the problem of the choice of local operations (being decided classically) being pseudorandom. Second, it has to be done to a large enough ensemble, and later on, the memory of which random rotation was applied to which pair has to be forgotten. Obtaining Bell diagonal states via twirling could thus potentially be a very time-consuming process. Our result, Eq. (17), clearly illustrates that one can produce a Bell diagonal state from any mixed state by local actions *without twirling*. One simply has to teleport the state $|\psi^+\rangle$ through the given mixed state using the standard teleportation protocol. Each member of the resultant ensemble is already in a Bell diagonal state without the necessity of forgetting any local actions. Moreover, the randomness is intrinsic "quantum" randomness, stemming from the teleportation protocol.

We pause here briefly to provide the generalization of our derivation to higher dimensional analogs of the standard teleportation scheme, with mixed resource states. For a $d \times d$ state system, the standard teleportation scheme is constructed using the maximally entangled state, $|\Psi^+\rangle = \frac{1}{\sqrt{d}} \sum_j |j\rangle |j\rangle$, and the set of unitary generators $U_{(1)}^{nm} = \sum_k e^{2\pi i k n/d} |k\rangle\langle k \oplus m|$, acting on the first part of the system, where $\oplus$ denotes addition modulo-d. The set of maximally entangled states is then denoted by $E^{nm} = U^{nm}|\Psi^+\rangle\langle\Psi^+|(U^{nm})^\dagger$, respectively, for $n, m = 0, 1, \ldots, d-1$. If steps corresponding to those of Eqs. (4)–(9) are carefully carried out in this case, the higher dimensional teleportation channel remains a depolarizing channel of the form

$$\Lambda \varrho = \sum_{nm} \mathrm{Tr}[E^{nm}\rho] U^{n(-m)} \varrho (U^{n(-m)})^\dagger. \tag{19}$$

Now we proceed to one of the most important consequences of our result, namely, the fact that the teleported state [Eqs. (16), (17), and (19)] is *always mixed,* apart from the isolated case of maximally entangled channel. This implies that the relative entropy between the input state and the output state will always be *finite*. This allows us to quantify the success of teleportation using the relative entropy. Without our result [Eqs. (16) and (19)], there is no way to be sure that relative entropy between the input and the output state of the standard teleportation protocol would not blow up. The quantum relative entropy [6,7,15] is defined as $S(\rho\|\omega) = \mathrm{Tr}[\rho \log\rho - \rho \log\omega]$ and has a statistical interpretation [16], where the probability of mistaking the state ω for the state ρ after N measurements is given by $P(\omega \to \rho) \simeq e^{-NS(\rho\|\omega)}$ as $N \to \infty$. The success of teleportation may then be given by

$$\mathcal{F} = \overline{S(\psi_{\mathrm{in}}\|\omega_{\mathrm{out}})}, \tag{20}$$

averaged over all pure input states, ψ_{in}, in a similar way to fidelity, and ω_{out} is the output state. Physically, this has significance when a third party wishes to verify a, possibly imperfect, teleportation between two untrusted parties. We define imperfect as meaning the teleporting parties share no entanglement. The probability of the third party being fooled by the imperfect teleportation scheme, for a large number of states N, is given by $e^{-NS(\psi_{\mathrm{in}}\|\omega_{\mathrm{out}})}$, even assuming the third party is making optimal generalized collective measurements over the N teleportations. The relative entropy thus provides an asymptotic (collective) measure of teleportation success compared to the "single shot" nature of the fidelity measure.

The above measure can be readily applied to demonstrate that teleporting one-half of a maximally entangled state is a better way to detect the presence of entanglement than teleporting a single d state system. Using the quantum relative entropy to examine the fidelity of entanglement swapping, we can choose the state $|\Psi^+\rangle$ as the input state, and the relative entropy, $\mathcal{F}^+$, is given by

$$S(|\Psi^+\rangle\langle\Psi^+|||\omega_{\text{out}}) = -\text{Tr}[|\Psi^+\rangle\langle\Psi^+|\log\omega_{\text{out}}] \quad (21)$$

$$= -\log\text{Tr}[|\Psi^+\rangle\langle\Psi^+|\rho] \quad (22)$$

$$= -\log F, \quad (23)$$

the negative log of the singlet fraction of the resource ρ. The maximally entangled fraction for separable states is bound by $1/d^2 \leq F \leq 1/d$, which gives bounds on the relative entropy,

$$\log d^2 \geq \mathcal{F}^+ \geq \log d. \quad (24)$$

Since Eq. (20) is bounded above by $\mathcal{F} \leq \log d$ (for teleportation of a single d state system), the optimal method for verification of the *presence of entanglement* through teleportation is by sending half of a maximally entangled $d \times d$ pair through the teleportation channel.

We now proceed to show how the success of teleportation, when quantified by the relative entropy, can be related to a classical capacity. The classical capacity of communication using the quantum states $\rho_i = \sigma_i \rho \sigma_i$ as letters (for qubits σ_i are the Pauli matrices and identity, while for higher dimensions, they are corresponding generalizations), with *a priori* probabilities p_i is given [8] by $C = \sum_i p_i S(\rho_i || \sum_j p_j \rho_j)$. From Eqs. (16) and (20), it is clear that each term in the above summation can be interpreted as an *unaveraged* relative entropy measure of success of a standard teleportation protocol with a different utilization of the same resource. The particular state to be teleported is ρ and the resource χ has maximally entangled components with weights p_i. While the first term corresponds to optimal utilization of the resource for teleportation, the other three terms correspond to a less efficient teleportation using the maximally entangled components of lower weight as the principal state ($|\Psi^+\rangle$) for teleportation. Worse teleportation implies a greater value of the relative entropy between the input and the output state, by virtue of which we have

$$\mathcal{F} \leq \overline{C}, \quad (25)$$

where $\overline{C}$ is the average of C taken over all possible pure input $\rho = \psi_{\text{in}}$. Physically, this means that the relative entropy measure of teleportation success will be bounded above by the average classical communication capacity using pure letter states related by Pauli rotations with *a priori probabilities being given by the weights of the maximally entangled components of the resource*. This result can be regarded as connecting a quantum and a classical capacity.

In this Letter we have presented an explicit expression for the output of a standard teleportation protocol using an arbitrary mixed resource. Most known results about the standard teleportation process [2,5] follow quite straightforwardly from our expression. It also has the potential for generating a host of other results (of which, we have given three distinct examples) relating to the standard teleportation process with an arbitrary mixed state. Most importantly, our result *allows* us to define a statistical measure of teleportation success in terms of relative entropy. It will be straightforward to generalize our result to multiparty scenarios of entanglement swapping [14] with Greenberger-Horne-Zeilinger state measurements and arbitrary mixed states. The use of "twisted" entangled states [17] may also lead to the generalization of this result to arbitrary teleportation schemes.

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