

Aging, Phase Ordering, and Conformal Invariance

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In a variety of systems which exhibit aging, the two-time response function scales as $R(t, s) \approx s^{-1-a} f(t/s)$. We argue that dynamical scaling can be extended towards conformal invariance, thus obtaining the explicit form of the scaling function f . This quantitative prediction is confirmed in several spin systems, both for $T < T_c$ (phase ordering) and $T = T_c$ (nonequilibrium critical dynamics). The 2D and 3D Ising models with Glauber dynamics are studied numerically, while exact results are available for the spherical model with a nonconserved order parameter, both for short-ranged and long-ranged interactions, as well as for the mean-field spherical spin glass.

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Aging phenomena are observed in a broad variety of systems with slow relaxation dynamics [1]. Aging behavior is known to be more fully revealed by two-time quantities, rather than by one-time quantities (see [2,3] for reviews). The most commonly studied two-time quantities are the correlation function $C(t, s) = \langle \phi(t)\phi(s) \rangle$ and the response function $R(t, s) = \delta \langle \phi(t) \rangle / \delta h(s)$, where ϕ is the order-parameter field at some point, and h is the conjugate local magnetic field, while s is the waiting time and t is the observation time.

Consider for definiteness, instead of a genuine glassy system, the situation of a ferromagnetic model, evolving at fixed temperature T from a disordered initial state. In the high-temperature phase ($T > T_c$), the relaxation time is small, so that the system relaxes rapidly to equilibrium, where $C(\tau)$ and $R(\tau)$ are time-translation invariant (they depend only on the difference $\tau = t - s$) and obey the fluctuation-dissipation theorem: $TR(\tau) = -dC(\tau)/d\tau$. In the low-temperature phase ($T < T_c$), where coarsening takes place, both $C(t, s)$ and $R(t, s)$ depend nontrivially on the ratio $x = t/s$ in the self-similar regime of phase ordering [4]. Similar features are also observed in the late stages of critical dynamics ($T = T_c$) [5–8]. The distance from equilibrium of an aging system is usually characterized by the fluctuation-dissipation ratio $X(t, s)$ [2,3,9,10], such that $TR(t, s) = X(t, s)\partial C(t, s)/\partial s$.

In the scaling regime where s and $\tau = t - s$ are simultaneously much larger than the microscopic time scale (set to unity), the scaling laws

$$C(t, s) \approx s^{-b} F(t/s), \quad R(t, s) \approx s^{-1-a} f(t/s) \quad (1)$$

are found to hold for a broad range of models [2–4]. Moreover, for $x \gg 1$, i.e., $1 \ll s \ll t$, both scaling functions usually fall off as

$$F(x) \sim f(x) \sim x^{-\lambda/z}, \quad (2)$$

where z is the dynamic critical exponent and λ is the auto-correlation exponent [11]. For a ferromagnetic model, with

scalar nonconserved order parameter, (1) holds at criticality ($T = T_c$) [5,8], with $a = b = 2\beta/(\nu z) = (d - 2 + \eta)/z$, where β , ν , η are static exponents, while z is known from equilibrium critical dynamics. In the phase-ordering regime ($T < T_c$), (1) holds only in the aging regime, where $C(t, s)$ decays from its plateau value $q_{EA} = M_{eq}^2$ to zero (M_{eq} is the spontaneous magnetization). One has $b = 0$ and $F(1) = q_{EA}$. There seems to be no general result for the response exponent: $a = 1/2$ for the Glauber-Ising model, both in one dimension [6,7] and in higher dimensions [12], while $a = d/2 - 1$ for the spherical model in dimension $d > 2$ [8,13,14]. We stress that the specific scaling forms (1) and (2) hold only for a completely *disordered* initial state. If there are correlations in the initial state, similar but different scaling forms hold [15,16].

At present, there is no general principle to predict the form of correlation and response functions in nonequilibrium systems. On the other hand, for an *equilibrium* critical point (where formally $z = 1$), scale invariance can be extended to conformal invariance [17–19]. Scale invariance implies that correlators transform covariantly under dilatations, i.e., scale transformations which are spatially uniform. Conformal transformations are *local* scale transformations with a position-dependent dilatation factor $b = b(\mathbf{r})$, but such that angles are conserved. This is already enough to fix the form of an equilibrium correlator at criticality in any space dimension. Furthermore, 2D conformal invariance yields exact values for the entire set of critical exponents, the exact form of all n -point correlators, a classification of the universality classes, and much more [20] (see [18,19] for reviews).

Is a similar extension of the scale invariance of (1) also available for nonequilibrium systems? Indeed, we present evidence that this might be so. First, we argue [21] that the two-time response function $R(t, s)$ should transform covariantly under the action of conformal transformations

in time. This assumption is then shown to imply the scaling form

$$R(t, s) \approx r_0(t/s)^{1+a-\lambda/z}(t-s)^{-1-a}, \quad (3)$$

i.e., $f(x) = r_0x^{1+a-\lambda/z}(x-1)^{-1-a}$.

The scaling function $f(x)$ is thus entirely fixed, up to the normalization constant r_0 , by the two exponents a and λ/z entering (1) and (2). This explicit scaling form (3) of the response function is the main result of this Letter. It is expected to hold throughout the aging regime, for any value of the ratio $x = t/s > 1$. The power law (2) is recovered for well-separated times ($x \gg 1$) [22].

The prediction (3) will be corroborated by numerical simulations in the 2D and the 3D Ising model with Glauber dynamics. In addition, exact results for the spherical model with a nonconserved order parameter (including spatially long-ranged interactions and/or quenched disorder) also reproduce (3) [5,8,10,13,15,23,24]. This confirmation provides, for the first time, evidence for conformal invariance in nonequilibrium and aging phenomena.

We now sketch the line of reasoning leading to (3). The full calculation will be given elsewhere [21]. To begin, we ask, ‘‘What space-time symmetries are consistent with dynamical scale invariance $t \rightarrow b^z t$, $\mathbf{r} \rightarrow b\mathbf{r}$, where z is the dynamical exponent [21,25]?’’ A similar question has already been successfully raised for equilibrium systems with strongly anisotropic scaling [25,26]. While in equilibrium systems the correlation functions (of quasiprimary operators [20]) are expected to transform in a simple way, for nonequilibrium systems it is rather the response functions which will take this role, as argued long ago [27]. We expect the requested extension of dynamical scaling to contain the Möbius transformations of time: $t \mapsto t' = (\alpha t + \beta)/(\gamma t + \delta)$, with $\alpha\delta - \beta\gamma = 1$, since these occur in the two known special cases, namely, conformal invariance for $z = 1$ and Schrödinger invariance for $z = 2$. It turns out that this condition is already sufficient to fix the form of the infinitesimal generators. In the one space dimension, to which we restrict ourselves for notational simplicity, it can be shown that for nonequilibrium systems one may write [21]

$$\begin{aligned} X_{-1} &= -\partial_t, & X_0 &= -t\partial_t - (1/z)r\partial_r, \\ X_1 &= -t^2\partial_t - (2/z)tr\partial_r - \beta r^2\partial_r^{2-z}, \end{aligned} \quad (4)$$

where β is a constant related to ‘‘mass’’ [21,25]. The generators X_n satisfy the commutation relations $[X_n, X_m] = (n-m)X_{n+m}$ (with $n, m \in \{-1, 0, 1\}$) of the Lie algebra of the conformal group.

Equation (4) forms the basis for the derivation of (3). Time translations are generated by X_{-1} . In order to make the above construction applicable to nonequilibrium situations, we must discard the latter and require only covariance under the subalgebra S generated by X_0 and X_1 (see [23]). It is clear from the form of the generators that the initial line $t = 0$ is invariant under the

action of S . Now, consider a general response function $G = \langle \phi_1(t_1, r_1) \tilde{\phi}(t_2, r_2) \rangle$, where the field ϕ_1 is characterized by its scaling dimension x_1 and mass β_1 , and the response field $\tilde{\phi}_2$ (see, e.g., [17]) has scaling dimension x_2 and mass β_2 . Then the covariance of G under the local scale transformations in S is expressed by the conditions [21] $X_0 G = (\zeta_1 + \zeta_2)G$, $X_1 G = (2\zeta_1 t_1 + 2\zeta_2 t_2)G$, where $\zeta_i = x_i/z$ ($i = 1, 2$). Moreover, we require spatial translation invariance; thus $G = G(t_1, t_2; r_1 - r_2)$. This is always satisfied if $\beta_1 + (-1)^{2-z}\beta_2 = 0$. We can now set $r = r_1 - r_2 = 0$ and obtain the response function $G = R(t_1, t_2)$. Then the generators (4) reduce to the standard conformal generators (see, e.g., [18–20]), and R satisfies the differential equations

$$\begin{aligned} (t\partial_t + s\partial_s + \zeta_1 + \zeta_2)R(t, s) &= 0, \\ (t^2\partial_t + s^2\partial_s + 2\zeta_1 t + 2\zeta_2 s)R(t, s) &= 0; \end{aligned} \quad (5)$$

hence $R(t, s) = r_0(t/s)^{\zeta_2 - \zeta_1}(t-s)^{-\zeta_1 - \zeta_2}$. An identification of exponents with (1) and (2) yields (3).

The prediction (3) will now be checked against results for various model systems. We begin with a novel numerical investigation of the Ising model, on square or cubic lattices, with periodic boundary conditions, and Glauber or heat-bath dynamics. Because the instantaneous response function $R(t, s)$ is too noisy to be measured in a simulation, we consider instead the integrated response function [8,28]

$$\rho(t, s) = T \int_0^s du R(t, u) \approx (T/h)M_{\text{TRM}}(t, s), \quad (6)$$

where $M_{\text{TRM}}(t, s)$ is the thermoremanent magnetization, i.e., the magnetization of the system at observation time t obtained after applying locally a small magnetic field h between the initial time $t = 0$ and the waiting time $t = s$. This quantity can be readily measured, either in TRM experiments or in numerical simulations [28]. The data shown in Figs. 1 and 2 have been obtained in this way and averaged over at least 1000 different realizations of systems with 300×300 spins in 2D and $50 \times 50 \times 50$ spins in 3D. Larger systems were also simulated, in order to check for finite-size effects. Table I contains the numerical values of exponents used in the subsequent analysis.

Figure 1 displays our results for $\rho(t, s)$ in the scaling regime at criticality (data corresponding to $1 \sim \tau \ll s$ have been discarded). From (1) we expect a data collapse if $s^a \rho(t, s)$ is plotted against $x = t/s$, and this is indeed the case. Having thus confirmed the expected scaling, we can compare with the prediction (3). We find complete quantitative agreement, after adjusting only one parameter, the overall normalization constant r_0 . Figure 2 shows our results in 2D and 3D, in the scaling regime and for a fixed temperature below T_c . As $a = 1/2$ [12], we expect that $s^{1/2} \rho(t, s)$ depends only on t/s . This is indeed the case in 3D, but for 2D the situation is more complicated. Analytical calculations in the spirit of the Ohta-Jasnow-Kawasaki approximation [4] reveal the presence of extra

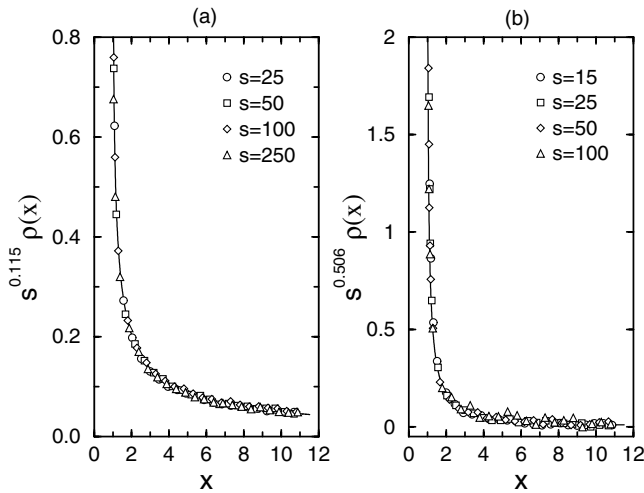


FIG. 1. Scaling of the integrated response function ρ for (a) the 2D and (b) the 3D Glauber-Ising model at criticality ($T = T_c$). The symbols correspond to different waiting times. The full curve is the conformal invariance prediction for ρ , obtained by integrating (3).

logarithms: $\rho(t, s) \approx s^{-1/2} \ln s f(t/s)$ [12]. Logarithmic corrections to scaling are not so rare, even in the realm of 2D equilibrium conformal theories [19,29]. We therefore propose the heuristic ansatz

$$\rho(t, s) \approx s^{-1/2} (r_0 + r_1 \ln s) E(t/s), \quad (7)$$

where r_0, r_1 are nonuniversal constants, and $E(x)$ is a scaling function. It is apparent from Fig. 2(a) that we thus obtain a satisfactory scaling. We have checked that the same scaling form holds in the entire low-temperature phase, where r_0, r_1 depend on T . We again find complete agreement between the form of the scaling function $E(x)$ and the prediction (3).

We now turn to confirmations of (3) by means of available analytical results. For the ferromagnetic spherical model [8], which can alternatively be described in terms of a continuum field theory [5,15], the scaling expression of the response function has been derived in any dimension $d > 2$. It reads $R(t, s) \approx (4\pi s)^{-d/2} f(x)$, where the scaling function is, in the ordered phase ($T < T_c$) [8,15],

$$f(x) = x^{d/4} (x - 1)^{-d/2}, \quad (8)$$

and at the critical point ($T = T_c$) [5,8],

TABLE I. Critical temperature and exponents of the 2D and 3D Glauber-Ising model, both in the phase-ordering regime ($T = 0$) and for critical dynamics ($T = T_c$).

		2D	3D
T_c		2.2692	4.5115
z	$T = 0$	2	2
	$T = T_c$	2.17	2.04
λ	$T = 0$	1.25	1.50
	$T = T_c$	1.59	2.78
β/ν	$T = T_c$	1/8	0.517

$$f(x) = \begin{cases} x^{1-d/4} (x - 1)^{-d/2} & (2 < d < 4). \\ (x - 1)^{-d/2} & (4 < d). \end{cases} \quad (9)$$

These expressions are in full agreement with (3). The second expression of (9), corresponding to the mean-field situation, coincides with the result for a free (Gaussian) field [10], as could be expected.

The spherical model has the peculiarity that the dynamical exponent $z = 2$ throughout. In that case, the response functions are expected to transform covariantly under the Schrödinger group [23]. The full space-time dependent response reads

$$\langle \phi(t, \mathbf{r}_1) \tilde{\phi}(s, \mathbf{r}_2) \rangle = R(t, s) \exp\left(-\frac{\mathcal{M}}{2} \frac{(\mathbf{r}_1 - \mathbf{r}_2)^2}{t - s}\right), \quad (10)$$

with $R(t, s)$ given by (3), and where the mass \mathcal{M} is a constant [in (4), $\beta = \mathcal{M}/2$ for $z = 2$]. A comparison with the exact spherical model results, both at and below T_c [5,15], also permits us to confirm this fully [19,23].

Recently, correlation and response functions have been calculated exactly for the spherical model with long-range interactions of the form $J(\mathbf{r}) \sim |\mathbf{r}|^{-d-\sigma}$ [24]. For $d > 2$ and $\sigma > 2$, the spherical model with short-ranged interactions, discussed above, is recovered. On the other hand, for $d > 2$ and $0 < \sigma < 2$, or $d \leq 2$ and $0 < \sigma < d$, the dynamical exponent reads $z = \sigma$ below criticality, while the response function scales as [24]

$$R(t, s) \approx r_0 (t/s)^{d/(2\sigma)} (t - s)^{-d/\sigma}, \quad (11)$$

which again agrees with (3). This example illustrates that spatially long-ranged interactions need not destroy conformal invariance in nonequilibrium situations, in contrast to the situation of conformal invariance *at* equilibrium. Moreover, for the mean-field spherical spin glass [13], the

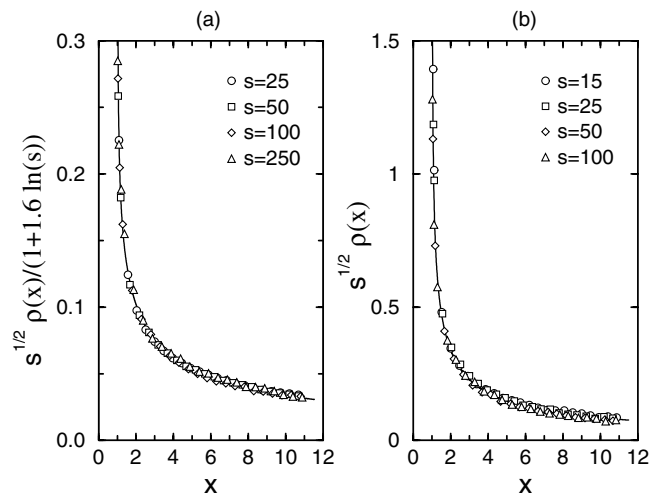


FIG. 2. Scaling of the integrated response function ρ for the low-temperature Glauber-Ising model (a) in 2D at $T = 1.5$ and (b) in 3D at $T = 3$. The symbols correspond to different waiting times. The full curves are obtained by integrating (3).

response function reads $R(t, s) \sim (t/s)^{3/4}(t-s)^{-3/2}$ in the low-temperature aging regime. This result also agrees with (3). It coincides with (8) for $d = 3$, as a consequence of the known similarity between the 3D spherical ferromagnet and the mean-field spin glass [14]. Finally, note that for the simple random walk $R(t, s) = r_0 = \text{const.}$ for $t > s$, see [10], which is consistent with (3) with exponents $a = -1$ and $\lambda/z = 0$.

In conclusion, the dynamical scale invariance realized in nonequilibrium aging phenomena apparently generalizes towards (a subgroup of) conformal invariance. As a first consequence, we obtained the explicit scaling expression (3) for the two-time response function $R(t, s)$, whose functional form depends only on the values of the exponents a and λ/z . Our prediction (3) has been checked against analytical and numerical results for several spin systems with a nonconserved order parameter. The entire evidence available at present comes from classical systems with a disordered initial state. Different initial conditions may lead to a modified scaling behavior [15,16] and the applicability of conformal invariance to these remains to be studied. The problem of identifying the full set of physical conditions on the systems which obey (3) remains open. However, the validity of (3) might extend to a broader class of systems than studied here, possibly including some realistic glassy systems. Finally, it appears that the conditions for the applicability of conformal invariance in nonequilibrium situations (such as the value of z or the presence of long-ranged interactions) are *less* restrictive than for conformal invariance at equilibrium critical points. We hope that the ideas presented might shed some light on some of the standing questions of aging phenomena [30].

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