Scaling of Collisionless Forced Reconnection

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The scaling of the reconnection electric field in a collisionless plasma is determined analytically for a model of forced reconnection. In particular, the dependence of the length of the reconnection layer on the ion skin depth and the boundary conditions is calculated explicitly. Analytical results are tested by Hall magnetohydrodynamics simulations.

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During the last four decades, two models of steady-state reconnection—Sweet-Parker [1,2] and Petschek [3]—have been the focal points of discussions on nonlinear reconnection dynamics. Both models were based on resistive magnetohydrodynamics (MHD). To fix ideas, let us consider a sheared magnetic field, $\mathbf{B} = B_P\hat{\mathbf{x}} + B_T\hat{\mathbf{y}} = B_{P0}\tanh(z/a)\hat{\mathbf{x}} + B_T\hat{\mathbf{y}}$, where B_{P0} and B_T are positive constants. The poloidal component of the magnetic field, B_P, changes sign across the so-called neutral line at $z = 0$. In the Sweet-Parker model, assuming that the plasma is incompressible, steady-state reconnection occurs in the vicinity of the neutral line on the time scale $\tau_{SP} \equiv (\tau_A\tau_R)^{1/2} = S^{1/2}\tau_A$ where $\tau_A \equiv a/v_A = a(4\pi\rho)^{1/2}/B_{P0}$ is the poloidal Alfvén time, $\tau_R \equiv 4\pi a^2/\eta c^2$ is the resistive diffusion time, and $S \equiv \tau_R/\tau_A$ is the Lundquist number. (Here ρ is the mass density, η is the resistivity of the plasma, and c is the speed of light.) The reconnection layer has the geometric structure of Y points [4], and its length is of the order of the system size. For weakly collisional systems such as the solar corona, the Lundquist number S is typically very large ($\sim 10^{12}$–10^{14}) and hence, the time scale τ_{SP} is of the order of hours. Since τ_{SP} is much too long to account for fast events such as solar flares, Petschek proposed a qualitatively different steady-state model which maintains an X-point structure for all times. In contrast with the Sweet-Parker model, Petschek's model yields a fast reconnection time scale, with a weak logarithmic dependence on S. For the high-S solar corona, the Petschek time scale is of the order of minutes, much closer to the relevant time scale for flares.

Since the mid-1980s, computer simulations of high-S plasmas have shown that even if one begins with an initial equilibrium state containing an X point that would appear to favor Petschek, one generally ends up obtaining an extended reconnection layer with Y-point structure typical of Sweet-Parker [5]. For high-S plasmas, this leaves us with a quandary. Whereas the Sweet-Parker time scale is dynamically realizable, it is much too slow. On the other hand, the Petschek model, which yields a faster and more physically relevant time scale, appears not to be realizable in the high-S regime.

In recent years, it has become clear that a possible resolution of this quandary may be found by going beyond the resistive MHD model and including collisionless effects via the generalized Ohm's law. High-S plasmas tend to develop thin and intense current sheets in the reconnection layer. As these thin current sheets become localized and intense and their width Δ_η falls in the range $d_e \equiv c/\omega_{pe} \ll \Delta_\eta \le d_i \equiv c/\omega_{pi}$ (where ω_{pe} and ω_{pi} are the electron and ion plasma frequencies, respectively), we need to replace the standard Ohm's law in resistive MHD by the generalized Ohm's law which can be written

$$\mathbf{E} + \mathbf{v} + \mathbf{B} = \frac{1}{S}\mathbf{J} + \frac{d_i}{n}(\mathbf{J}\times\mathbf{B} - \beta_{ep}\nabla p), \quad (1)$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, $\mathbf{v}$ is the plasma flow velocity, $\mathbf{J}$ is the current density, and p is the electron pressure (assumed to be a scalar). In (1), we have redefined the following variables (expressed in cgs units) to make them dimensionless: $c\mathbf{E}/(B_{p0}V_A) \to \mathbf{E}$, $\mathbf{B}/B_{p0} \to \mathbf{B}$, $\mathbf{v}/V_A \to \mathbf{v}$, $a\nabla \to \nabla$, $4\pi a\mathbf{J}/(cB_{p0}) \equiv \mathbf{J}/J_0 \to \mathbf{J}$, $p/(n_0T_e) \to p$, $n/n_0 \to n$, $d_i/a \to d_i$, $d_e/a \to d_e$, $\beta_{ep} \equiv 4\pi n_0T_e/B_{p0}^2$. (Here n_0 is the average ion and electron density in a hydrogen plasma.) The second term on the right of (1) is proportional to the Hall current and the third to the electron pressure gradient. In recent years, the influence of the electron pressure gradient and Hall current on nonlinear reconnection dynamics has been the focus of extensive research in fusion [6–12], as well as space physics [13–22].

The main purpose of the present Letter is to present a scaling analysis of quasisteady collisionless forced reconnection within the framework of the generalized Ohm's law (1). While our results are qualitatively consistent with recent numerical simulations and scaling analyses, they go well beyond presently known results. Figure 1 shows a typical flux-surface plot from a nonlinear Hall MHD simulation under quasisteady conditions, with a reconnection layer of length ℓ and width Δ. Assuming that the plasma density is approximately constant, the inflow velocity v_{in}, which determines the reconnection rate, can be simply obtained from mass conservation. It is given by the expression $v_{\text{in}} \approx (\Delta/\ell)v_{\text{out}}$, where v_{out} is the outflow velocity.

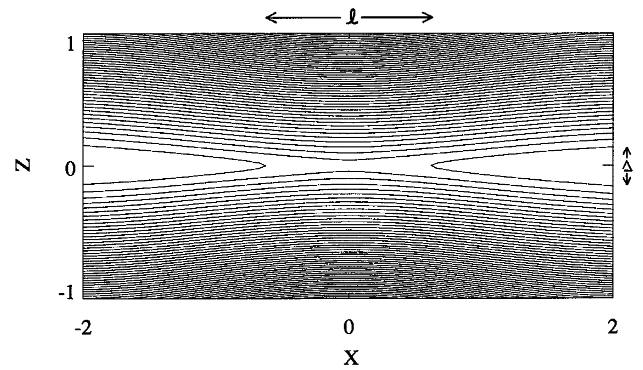

FIG. 1. Flux surface plot on the (x, z) plane from a Hall MHD simulation indicating schematically the width Δ and length ℓ of the reconnection layer.

On the basis of analysis and simulations, there are good estimates for Δ and v_{out} in the literature [9,16,19,22]. For example, when the equilibrium guide field is zero, it can be shown that $\Delta \approx d_i/\sqrt{2}$ [22], while v_{out} is equal to the Alfvén speed in the asymptotic outflow region. However, there is no theory that calculates the important length parameter ℓ from first principles. While ℓ is observed in several simulations to be significantly smaller than the system size, its parametric dependence on local parameters (such as the ion skin depth) or global parameters (such as the wave number of the boundary perturbation) is not known. On the basis of numerical simulations, it has been claimed [23] that the reconnection rate is a "universal constant" and given by $v_{\text{in}} \approx 0.1V_A$, which corresponds to $\ell \approx 10d_i$. If true, such a claim would imply that the collisionless reconnection rate is independent of global conditions, such as the form and structure of the boundary perturbations. Elsewhere, we have presented numerical evidence that questions this claim [22]. In this Letter, we attempt to settle this question by determining analytically the dependency of the parameter ℓ on local and global parameters for a model of forced reconnection. We also test the analytical scaling by simulations using the University of Iowa (UI) Hall MHD code.

To determine ℓ, we need to represent analytically the geometry of the reconnected flux surfaces. Since y is an ignorable coordinate for all times, we can write $B_P = \nabla\psi(z,x) \times \hat{\mathbf{y}}$. If reconnection is forced by a sinusoidal perturbation of wave number k, the magnetic flux function in a quasisteady state can be written $\psi(z,x) = \psi_0(z) + \tilde{\psi}(z)\cos kx$, where $\psi_0(z) = z^2/2$ is the equilibrium flux near the neutral line. We assume that the flux surface has the geometry of an X point. In the vicinity of the X point $(x, z \ll 1)$, we can then write

$$\psi(x,z) \approx \frac{1}{2}z^2 + \tilde{\psi}(0)\left(1 - \frac{k^2x^2}{2}\right) = \frac{1}{2}z^2 + \frac{w^2}{2}\left(1 - \frac{k^2x^2}{2}\right), \qquad (2)$$

where w is the island width, related to $\tilde{\psi}(0)$ by the relation $w \approx ((2\tilde{\psi}(0))^{1/2}$.

From (1), the reconnection electric field at the inner limit of the outer (or ideal region) is given by

$$\frac{\partial\psi}{\partial t} \approx v_{\text{in}}B_R \approx \frac{\Delta}{\ell}v_{\text{out}}B_R \approx \frac{\Delta}{\ell}B_R^2, \qquad (3)$$

where B_R is the magnitude of the poloidal field at the inner (outer) limit of the ideal (reconnection) region and $v_{\text{out}} \approx v_A = B_R$ (in dimensionless variables). Downstream of the reconnection layer (along x), we have

$$\frac{\partial\psi}{\partial t} \approx -v_x\frac{\partial\psi}{\partial x}\bigg|_{x\sim\ell/2} \approx \frac{1}{4}k^2w^2\ell B_R. \qquad (4)$$

Matching (3) and (4), we obtain

$$\ell \approx \frac{2\sqrt{B_R\Delta}}{kw}. \qquad (5)$$

As demonstrated elsewhere [22], both with and without a guide field, $\Delta \propto d_i$, which implies that $\ell \propto d_i^{1/2}$.

To complete the calculations, we now need to calculate the parametric dependencies of B_R and w on boundary conditions. For specificity, we consider an example of forced reconnection where the equilibrium is driven by inward flows of the form [22]

$$\mathbf{v}(x, z = \pm a, t) = \mp\hat{\mathbf{z}}V(t)(1 + \cos kx) = \mp\hat{\mathbf{z}}V_0\tanh\left(\frac{t}{\tau}\right) \times \left[1 - \tanh\left(\frac{t - t_0}{\tau}\right)\right](1 + \cos kx), \qquad (6)$$

imposed at the upper and lower boundaries. The imposition of inward flows for a finite time (of order t_0) and their subsequent switch-off ensures that the reconnection rate attains a maximum, and then decays in time. The rapid turning on and switching off of the inward boundary flows in the ideal region, where field lines are frozen in the plasma, deform the equilibrium boundaries located at $z = \pm a$. The perturbed boundaries are also flux surfaces and in the limit of large time given by

$$z \approx a \pm (1 + \cos kx)\int_0^\infty V(t)\,dt = a \pm \delta_0(1 + \cos kx) \quad \text{where } \delta_0 \equiv \int_0^\infty V(t)\,dt.$$

With these boundary conditions, the perturbed flux in the quasisteady outer region can be obtained by linearizing the force balance condition $\nabla \times (\mathbf{J} \times \mathbf{B}) = 0$. We obtain, in the approximation $\delta_0 \ll 1$ [24,25],

$$\tilde{\psi}(z) \approx \tilde{\psi}(0)\left[\cosh kz - \frac{\sinh|kz|}{\tanh k}\right] + \frac{\delta_0\sinh|kz|}{\sinh k}. \qquad (7)$$

Using the results developed in [26], it is also easy to show that $B_R \approx 2(1 + k/\sinh k)\delta_0$, which makes explicit the dependence of B_R on boundary conditions. In what follows, we choose the dimensionless values $V_0 = 0.005$, $\tau = 2$, $k = \pi/2$, and $t_0 = 4$. With these parameters, we obtain $\delta_0 \approx 0.028$ and $B_R \approx 0.093$.

The reconnection parameter Δ' is given by

$$\Delta' = \frac{\tilde{\psi}'(z \to 0+) - \tilde{\psi}'(z \to 0-)}{\tilde{\psi}(0)} \approx -\frac{2k}{\tanh k} + \frac{2k\delta_0}{\tilde{\psi}(0)\sinh k}. \quad (8)$$

When the reconnection rate is stationary, from the condition $\Delta' = 0$, we obtain the perturbed flux

$$\tilde{\psi}(0) \approx \frac{\delta_0}{\cosh k}. \quad (9)$$

From (9) and the expression for the island width $w \approx [2\tilde{\psi}(0)]^{1/2}$, we obtain

$$\ell \approx \frac{2}{k}\sqrt{\frac{\Delta(k + \sinh k)}{\tanh k}}. \quad (10)$$

In the absence of an equilibrium guide field, that is, $B_T = 0$, the width of the reconnection layer is estimated to be $\Delta \approx d_i/\sqrt{2}$, and the reconnection electric field is given by [22]

$$E_y = \frac{\partial \psi}{\partial t} \approx v_{\text{in}} B_R \approx \frac{d_i B_R^2}{\sqrt{2}\,\ell}. \quad (11)$$

From (10) and (11) we see that the reconnection electric field is proportional to $\sqrt{d_i}$. Furthermore, the constant of proportionality is not a universal constant but actually determined by the boundary conditions through the parameters B_R, k, and δ_0. The same conclusion holds in the presence of the equilibrium guide field.

Equation (11) shows that the leading-order reconnection rate is independent of resistivity, which is a weaker dependence on resistivity than the Petschek model where the dependence is logarithmic. This can be understood by considering the spatial structure of the reconnection layer, which is composed of an inner resistive region and an intermediate Hall region. In the Hall region $\Delta_\eta < |z| \leq d_i$, we obtain [22]

$$\frac{\partial \psi}{\partial t} \approx d_i(\mathbf{J} \times \mathbf{B})_y \approx \frac{d_i}{\ell} b B_R \approx \frac{d_i}{\sqrt{2}\,\ell} B_R^2, \quad (12)$$

where $B_y = B_T + b(x,z)$ and $b(x,z)$ is the component of the guide field spontaneously generated by the Hall current. In the resistive region $|z| < \Delta_\eta$ which supports the current density J_y, $b(x,z)$ tends to zero as we approach the X point, and the resistive term dominates so that

$$\frac{\partial \psi}{\partial t} \approx \frac{J_y}{S} \sim \frac{B_R}{S\Delta_\eta}. \quad (13)$$

Matching the resistive layer solution (13) to the Hall layer solution (12), we obtain

$$\Delta_\eta \approx \frac{\sqrt{2}\,\ell}{S d_i B_R}. \quad (14)$$

We thus find that the width of the current sheet in Hall MHD reconnection scales as S^{-1} in contrast with that in the Sweet-Parker model where it scales as $S^{-1/2}$. As the quantity $S\Delta_\eta$ is independent of S to leading order, so is the reconnection electric field $\partial\psi/\partial t$. Figure 2 shows plots of $S^{-1}J_y$, $d_i(\mathbf{J} \times \mathbf{B})_y$, and E_y from the UI Hall MHD code as a function of z under quasistationary conditions, when the reconnection electric field attains its maximum value. These plots demonstrate that while the reconnection electric field at the X point is supported entirely by the resisitive diffusion term, it is the Hall current term (that matches with the resistive term) in order to support E_y over the broader reconnection layer.

We now present numerical tests of the analytically predicted scaling $E_y \propto d_i^{1/2}$ using the UI Hall MHD code. For a number of reasons, this test is not as easy as may seem at first glance. First, it is important to obtain numerical results over a sufficient range of d_i in order to establish scaling. Second, one expects that the agreement with analytical scaling should improve as d_i decreases, but this is also the regime in which the reconnection rate decreases. For very small values of d_i, the effects of numerical diffusion can mask weak collisionless effects, making it difficult to establish scaling. Furthermore, the system takes much longer to work its way through transients for smaller values of d_i before it can attain a quasistationary state, and longer runs are more likely to be polluted by the effects of numerical diffusion when collisionless effects

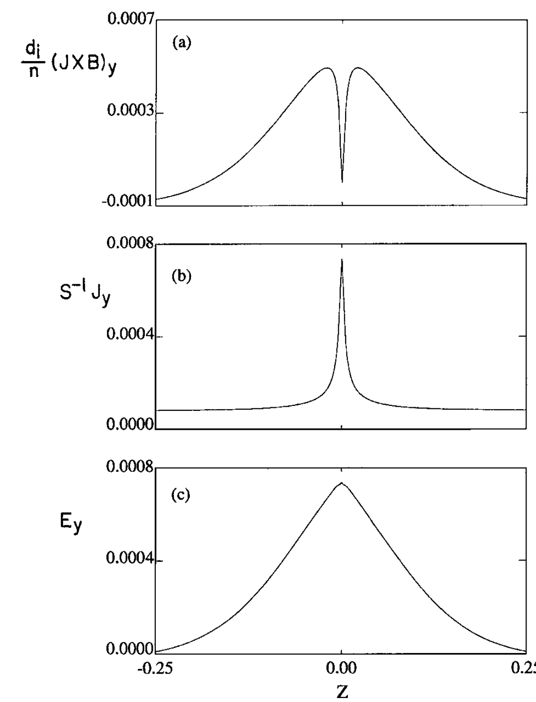

FIG. 2. Plots of (a) the Hall current term, (b) the resistive diffusion term, and (c) E_y in the generalized Ohm's law from a Hall MHD simulation as a function of z, with the neutral line at $z = 0$.

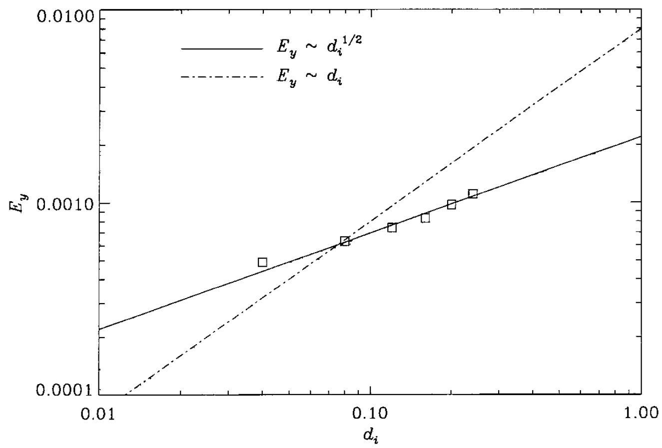

FIG. 3. Plot of E_y in a quasistationary state as a function of d_i, demonstrating that it scales as $d_i^{1/2}$. The numerical results are represented by squares.

are weak. Keeping these caveats in mind, we present in Fig. 3 a plot of the maximum value of E_y as a function of d_i, other parameters remaining the same as given earlier in this paper. The plot shows that the reconnection electric field scales as $\sqrt{d_i}$, as predicted by the analytical relations (10) and (11). In particular, for $d_i = 0.12$ and the parameters given above we obtain $\Delta \approx 0.085$, $\ell \approx 0.76$, and $\partial\psi/\partial t \approx 1.0 \times 10^{-3}$ which is slightly higher than the numerical result $\partial\psi/\partial t \approx 0.74 \times 10^{-3}$. The analytically predicted dependence of the reconnection rate on the wave number of the boundary perturbation is tested by varying the wave number k. For $k = \pi/2$, $\pi/3$, and $\pi/4$, the analytical predictions are $\partial\psi/\partial t \approx (1.0, 1.03,$ and $0.91) \times 10^{-3}$ which are also slightly higher than the numerical results given, respectively, by $\partial\psi/\partial t \approx (0.74, 0.86,$ and $0.70) \times 10^{-3}$. Following the development in [22], these results can be extended in a straightforward way to the case $B_T \neq 0$. It is interesting to note that the dependence of E_y on $d_i^{1/2}$ is the same as that obtained in our earlier analytical work on free reconnection due to the $m = 1$ kink-tearing instability [7].

In conclusion, we have given a scaling analysis of forced reconnection within the framework of Hall MHD when resistivity is the mechanism breaking field lines. In particular, we have obtained an explicit analytical expression for the length of the reconnection layer, which is much smaller than the system size and identified its dependencies on local plasma parameters as well as global parameters determined by boundary conditions. The analytical scaling relations have been tested by simulations using the UI Hall MHD code.

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