Aligning Reference Frames with Quantum States

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We analyze the problem of sending, in a single transmission, the information required to specify an orthogonal trihedron or reference frame through a quantum channel made out of N elementary spins. We analytically obtain the optimal strategy, i.e., the best encoding state and the best measurement. For large N, we show that the average error goes to zero linearly in $1/N$. Finally, we discuss the construction of finite optimal measurements.

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Can a system of N elementary spins be used to communicate in a single transmission the orientation of three mutually orthogonal unit vectors (orthogonal trihedron)? A positive answer would, e.g., enable two distant parties (Alice and Bob) to establish a common reference frame using just a quantum channel. This question was addressed 20 years ago by Holevo [1], who concluded that if such a quantum system has a well-defined total spin J the best the sender (Alice) can attempt to achieve is to transmit the orientation of *at most one* of the three vectors. There has recently been renewed interest in this simpler, more manageable, problem of sending a single direction, and reformulations and extensions of the original question abound in the literature [2–10] (related issues can also be found in [11]). In all the cases, optimal communication involves collective (entangled) measurements and an accurate choice of the messenger quantum states.

In this Letter, we will be concerned with the more complex problem of sending the information that specifies an orthogonal trihedron (OT). We will demonstrate that by encoding the relevant geometrical information in a particular class of states one overcomes the limitations foreseen by Holevo and a good transmission is possible. These states can be written as a simple superposition of states belonging to each of the SU(2) irreducible representations that build up the Hilbert space of the N spins. They have maximal third component of the total spin within each representation, i.e., in standard notation they are of the form $\sum_j C_j | j, m = j\rangle$ (therefore they are *not* eigenstates of either $\vec{J}^2$ or J_z). The quality of the optimal communication strategy is shown to increase with N and in the limit $N \to \infty$ the average error $\langle h\rangle$ goes to zero. For large N we obtain an analytical estimate of this error, $\langle h\rangle \gtrapprox 8/N$. We would like to emphasize that despite the apparent difficulty of the problem [12], an analytical treatment is possible, which provides us with a physical insight of the underlying quantum aspects involved in the communication process.

Let us suppose Alice has a system of N spins which she wishes to use to tell Bob an OT, $\mathsf{n} = \{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$. By performing quantum measurements, Bob will be able to reconstruct this OT with some accuracy and will make the guess $\mathsf{n}' = \{\vec{n}_1', \vec{n}_2', \vec{n}_3'\}$. The obvious parametrization of the different OT's is provided by the Euler angles α, β, γ, of the rotations that map $\mathsf{n}_0 = \{\vec{x}, \vec{y}, \vec{z}\}$ into n and n'. We will use g as a shorthand for the three Euler angles, i.e., $g = (\alpha, \beta, \gamma)$. Following Holevo [1], we quantify the quality of the communication strategy by evaluating the mean value of the error (or average error) defined for each individual measurement by

$$h(g, g') = \sum_{a=1}^{3} |\vec{n}_a - \vec{n}_a'|^2 = \sum_{a=1}^{3} |\vec{n}_a(g) - \vec{n}_a(g')|^2. \tag{1}$$

Assuming the OT's are chosen from an isotropic distribution, and denoting by $p_{g'}(g)$ the conditional probability of Bob guessing $\mathsf{n}(g')$ if Alice's OT is $\mathsf{n}(g)$, one has

$$\langle h\rangle = \int dg \int dg' h(g, g') p_{g'}(g), \tag{2}$$

where dg is the Haar measure of the rotation group, SU(2), which in terms of the Euler angles reads $dg = \sin\beta d\beta d\alpha d\gamma / 8\pi^2$. Covariance implies that (2) can be written as

$$\langle h\rangle = \int dg h(g, \mathbf{0}) p_{\mathbf{0}}(g), \tag{3}$$

where $\mathbf{0}$ stands for $(\alpha, \beta, \gamma) = (0, 0, 0)$. One can easily check that

$$h(g, \mathbf{0}) = 6 - 2 \operatorname{tr} U^{(1)}(g), \tag{4}$$

where $U^{(j)}$ is the SU(2) irreducible representation of spin j, whose elements are written as $\mathfrak{D}^{(j)}_{mm'}(g) = \langle j, m | U^{(j)}(g) | j, m'\rangle$. One also has $t \equiv \operatorname{tr} U^{(1)}(g) = \sum_m \mathfrak{D}^{(1)}_{mm}(g) = \cos\beta + (1 + \cos\beta)\cos(\alpha + \gamma)$. We see that the values of t lay in the real interval $[-1, 3]$. The value $t = 3$ corresponds to perfect determination of Alice's OT and implies that $h = 0$. Note also that $\langle h\rangle = 6 - 2\langle t\rangle$. Random guessing implies $\langle t\rangle = 0$ ($\langle h\rangle = 6$), while perfect determination of one axis and random guessing of the remaining two yield $\langle t\rangle = 1$ ($\langle h\rangle = 4$).

The most general quantum state Alice can use has the form $|A(g)\rangle = U(g)|A\rangle$. Here $U(g) = \bigoplus_j U^{(j)}$ and

$$|A\rangle = \sum_j |A^j\rangle = \sum_{j,m} A^j_m |j,m\rangle; \qquad \sum_{j,m} |A^j_m|^2 = 1, \quad (5)$$

where, for N even (odd), j runs from 0 (1/2) to $N/2$ (for simplicity we will only consider N even unless otherwise stated) [13], and m runs from $-j$ to j. $|A\rangle$ is a fixed reference state associated with the OT n_0.

Likewise, we may write a reference state $|B\rangle$ from which we can construct the projectors of Bob's positive operator valued measurement (POVM). The general form of the state is

$$|B\rangle = \sum_j \sqrt{2j+1}\,|B^j\rangle; \qquad |B^j\rangle = \sum_m B^j_m |j,m\rangle, \quad (6)$$

where the square root is introduced for later convenience, and the projectors are $O(g) = U(g)|B\rangle\langle B|U^\dagger(g)$. (This approach is fully covariant [1]. See [14] for a very recent discussion on noncovariant approaches.) We will first consider continuous POVM's for simplicity, but finite ones can also be constructed, as will be explained below. The condition $\mathbf{1} = \int dg\, O(g)$ requires that

$$\sum_m |B^j_m|^2 = 1, \quad \forall\, j, \quad (7)$$

as can be easily shown with the help of the orthogonality relations

$$\int dg\, \mathfrak{D}^{(j)}_{mm'}(g)\mathfrak{D}^{(l)*}_{nn'}(g) = \frac{\delta^{jl}\delta_{mn}\delta_{m'n'}}{2j+1}. \quad (8)$$

Quantum mechanics tells us that $p_{\mathbf{0}}(g) = |\langle B|U(g)|A\rangle|^2$, hence we have

$$\langle t\rangle = \int dg |\langle B|U(g)|A\rangle|^2\, \mathrm{tr}U^{(1)}(g). \quad (9)$$

In terms of the components of $|A\rangle$ and $|B\rangle$ the last expression reads

$$\langle t\rangle = \sum_{jl\cdots} \sqrt{(2l+1)(2j+1)}\, A^{l*}_n A^j_m B^l_{n'} B^{j*}_{m'} M^{lj}_{nmn'm'}, \quad (10)$$

where the sum is over all indices,

$$M^{lj}_{nmn'm'} = \int dg\, \mathrm{tr}U^{(1)}(g)\mathfrak{D}^{(j)}_{m'm}(g)\mathfrak{D}^{(l)*}_{n'n}(g) = \sum_M \langle 1Mjm\,|\,ln\rangle\langle 1Mjm'\,|\,ln'\rangle, \quad (11)$$

and the last terms in brackets are the usual Clebsch-Gordan coefficients.

The optimal strategy is the one that maximizes $\langle t\rangle$. It is tempting to introduce Lagrange multipliers λ and μ^j for the normalization constrains (5) and (7), respectively, and follow the standard maximization procedure. Analytical results along this line seem hard to obtain [12]. We will, thus, try to develop a more physical picture of Eqs. (9)–(11) which will lead us to a stunning simplification of the problem.

Notice that Eqs. (9)–(11) can also be written in a compact form as

$$\langle t\rangle = \sum_{lj} \frac{\sqrt{(2l+1)(2j+1)}}{3} \langle B^j\tilde{B}^l|P_1|A^j\tilde{A}^l\rangle, \quad (12)$$

where $|A^j\tilde{A}^l\rangle = |A^j\rangle \otimes |\tilde{A}^l\rangle$, the state $|\tilde{A}^j\rangle$ is the time reversed of $|A^j\rangle$, i.e., $\tilde{A}^j_m = (-1)^m A^{j*}_{-m}$ (and similarly for $|B^j\tilde{B}^l\rangle$ and $|\tilde{B}^l\rangle$) and P_1 is the projector over the Hilbert space of the representation of total spin $J = 1$. Our aim is to compute

$$\langle t\rangle_{\max} = \max_{AB}\langle t\rangle, \quad (13)$$

where the maximization is over all A^j_m and B^j_m subject to the normalization conditions in (5) and (7). The Schwarz inequality implies

$$\langle B^j\tilde{B}^l|P_1|A^j\tilde{A}^l\rangle \le \|P_1\,|A^j\tilde{A}^l\rangle\|\,\|P_1\,|B^j\tilde{B}^l\rangle\|, \quad (14)$$

where the equality holds iff

$$P_1|A^j\tilde{A}^l\rangle = \mu^{jl}P_1|B^j\tilde{B}^l\rangle, \qquad \forall\, j,l. \quad (15)$$

Hence, to compute $\langle t\rangle_{\max}$, we can restrict ourselves to a smaller parameter space, where $|A^j\rangle$ and $|B^j\rangle$ are constrained through (15). This is equivalent to consider only states $|A\rangle$ such that

$$A^j_m = C^j B^j_m, \qquad \text{with} \quad \sum_j |C^j|^2 = 1, \quad (16)$$

i.e., we need to consider only the set of parameters $\{C^j, B^j_m\}$. This we can prove, e.g., by induction on j using (15) with $l = j+1$ and starting with the trivial case $j = 0$. Equation (16) is easy to understand from the physical point of view. It just tells us that, for an optimal communication, the messenger states $|A(g)\rangle$ must be as similar as possible to the states $|B(g)\rangle$ on which the measuring device projects [7]. We next substitute (16) back into (12) to obtain

$$\langle t\rangle_{\max} = \max_{BC}\sum_{jj'} C^j \mathsf{M}^{jj'}_B C^{j'}, \quad (17)$$

where

$$\mathsf{M}^{jj'}_B = \frac{\sqrt{(2j+1)(2j'+1)}}{3}\langle B^j\tilde{B}^{j'}|P_1|B^j\tilde{B}^{j'}\rangle, \quad (18)$$

and the maximization is over all B^j_m and C^j subject to the normalizations (7) and (16).

Let us now discuss some properties of the matrix M_B defined by (18). We first note that M_B is tridiagonal, i.e., $\mathsf{M}^{jj'}_B = 0$ if $|j-j'| > 1$, and symmetric. It is manifestly non-negative, i.e., $\mathsf{M}^{jj'}_B \ge 0$ for all j, j' and, most important, it is rotationally invariant: any reference state of the form $|B'\rangle = U(g)|B\rangle$ is equally as good as $|B\rangle$.

We next compute bounds for the diagonal (M_B^{jj}) and off diagonal (M_B^{jj+1}) entries of M_B. We have

$$\mathsf{M}_B^{jj} = \frac{2j+1}{3}|\langle B^j \tilde{B}^j | 10\rangle|^2 \leq \frac{2j+1}{3}\left(\sum_{m'}|B^j_{m'}|^2\right)^2 \times |\max_m \langle jmj-m|10\rangle|^2 = \frac{j}{j+1} \equiv \mathsf{M}_{\rm op}^{jj}, \qquad (19)$$

where we have used rotational invariance to orient the (real) vector $P_1|B^j\tilde{B}^j\rangle$ along the z ($m=0$) axis. As for the off diagonal entries, the Schwarz inequality leads to

$$\mathsf{M}_B^{jj+1} \leq \frac{\sqrt{(2j+1)(2j+3)}}{3}\sum_{m'}|B^j_{m'}|^2 \times \sum_{m''}|\tilde{B}^{j+1}_{m''}|^2 \max_m\left(\sum_M \langle jM-mj+1m|1M\rangle^2\right) = \sqrt{\frac{2j+1}{2j+3}} \equiv \mathsf{M}_{\rm op}^{jj+1}, \qquad (20)$$

where, actually, the sum over M in the second line is independent of m. It is straightforward to verify that the particular choice

$$|B_{\rm op}\rangle = \sum_j \sqrt{2j+1}\,|j,j\rangle \Leftrightarrow B^j_{{\rm op}m} = \delta^j_m \qquad (21)$$

saturates the two upper bounds (19) and (20) *simultaneously*. Hence

$$\mathsf{M}_B^{jj'} \leq \mathsf{M}_{B_{\rm op}}^{jj'} \equiv \mathsf{M}_{\rm op}^{jj'}, \qquad (22)$$

for all j, j' and $|B\rangle$, where the nonvanishing entries of the matrix $\mathsf{M}_{\rm op}$ are defined in (19) and (20).

We now go back to (17) and compute $\langle t\rangle_{\max}$. We first note that, $\langle t\rangle_{\max} = \max_B \lambda(B)$, where $\lambda(B)$ is the maximal eigenvalue of the matrix M_B. Since M_B is non-negative, Eq. (22) implies [15]

$$\langle t\rangle_{\max} = \max_B \lambda(B) = \lambda(B_{\rm op}) \equiv \lambda_{\rm op}. \qquad (23)$$

We thus have simplified the problem to that of computing $\lambda_{\rm op}$, the maximal eigenvalue of $\mathsf{M}_{\rm op}$. This can be done proceeding along the same lines as in [7,9]. We would like to emphasize that the calculation relies on the fact that the maximal value of each entry of M_B is reached simultaneously, e.g., for the single state $|B_{\rm op}\rangle$. This is, *a priori,* a rather unexpected property which, however, provides a remarkable simplification of the calculation.

The result obtained and the form of the optimal state $|B_{\rm op}\rangle$ agree with our physical intuition as we now briefly discuss. If Alice's state has a well-defined total spin (i.e., it is an eigenstate of $\vec{J}^2$), $\mathsf{M}_{\rm op}$ becomes diagonal and $\langle t\rangle_{\max} = J/(J+1) = N/(N+2)$. In terms of the average error, $\langle h\rangle = 4(N+3)/(N+2)$, thus, at most ($N\to\infty$) $\langle h\rangle = 4$. In average, Bob cannot determine more than just one axis of Alice's trihedron. The structure of the state $|B_{\rm op}\rangle$ is such that, within each irreducible representation, the determination of a single axis is optimal [2] (this is *the best* Alice could do if she only were allowed to use a single irreducible representation). At the same time, $|B_{\rm op}\rangle$ is as different of an eigenstate of J_z as it can possibly be (if $|B_{\rm op}\rangle$ were an eigenstate of J_z, Alice would be able to communicate *only* a single axis).

For small N, one can easily obtain analytic expressions for $\langle t\rangle_{\max}$ (see Table I). For large N it suffices to give simple lower and upper bounds for $\langle t\rangle_{\max}$. A useful upper bound is provided by the condition $\langle t\rangle_{\max} \leq \max_j \sum_{j'} \mathsf{M}_{\rm op}^{jj'}$. A lower bound is obtained by computing $\Delta = \sum_{jj'} C^j \mathsf{M}_{\rm op}^{jj'} C^{j'}$ for any normalized vector with components C^j. A judicious choice is $C^j \propto \sqrt{2j+1}\,(N/2-j)(j+1)^p$. The maximum of Δ occurs at $p \approx \sqrt[3]{3N/4}$. We obtain

$$3 - \frac{4}{N} + O(N^{-4/3}) \lessapprox \langle t\rangle_{\max} \lessapprox 3 - \frac{4}{N} + O(N^{-2}). \qquad (24)$$

It is now clear that perfect determination of the trihedron, $\langle t\rangle_{\max} = 3$, is reached in the asymptotic limit, and it approaches three linearly in $1/N$. Finally, we have also performed a linear fit for $2000 \leq N \leq 7000$, and we have obtained

$$\langle t\rangle_{\max} \sim 3.0 - \frac{4.0}{N} - \frac{9.4}{N^{4/3}} + \ldots, \qquad (25)$$

which is completely consistent with (24).

We now turn our attention to the construction of POVM's with a finite number of outcomes, as they are the only ones that can be physically realized. The main idea is stated in [10] (see also [12]). We need to find a finite set $\{g_r\}$, $r = 1,\ldots,N(J)$, of elements of SU(2) and positive weights $\{c_r\}$ such that the orthogonality relation

$$\sum_{r=1}^{N(J)} c_r \mathfrak{D}^{(j)}_{mm'}(g_r)\mathfrak{D}^{(l)*}_{nn'}(g_r) = C_J \frac{\delta^{jl}\delta_{mn}\delta_{m'n'}}{2j+1}, \qquad (26)$$

holds for all $j, j' \leq J$, where $C_J = \sum_{r=1}^{N(J)} c_r$. This discrete version of (8) is valid only up to a maximal value J. The larger J is, the larger the $N(J)$ that must be chosen. Working along the same lines as in [10], one can show that it is sufficient to consider the $2J+1$ values $2\pi n/(2J+1)$, $n = 0,1,\ldots,2J$, for each one of the two angles α and γ. With this choice, Eq. (26) holds if the sets of values $\{\beta_r\}$ and $\{c_r\}$ are required to satisfy

$$\sum_r c_r P_L(\cos\beta_r) = 0, \qquad 1 \leq L \leq 2J, \qquad (27)$$

where P_L is the Legendre polynomial of degree L. A procedure to solve this system of equations is as follows. For J integer, define $\beta_r = \pi r/(2J+1)$, $r = 0,1,\ldots,2J+1$ and set $c_0 = c_{2J+1} = 1$. Note that $x_r \equiv \cos\beta_r$, $r =$

TABLE I. Maximal value of $\langle t\rangle$ vs the number of spins.

N	2	3	5	20	100	200
$\langle t\rangle_{\max}$	$\frac{3+\sqrt{57}}{12}$	$\frac{14+\sqrt{466}}{30}$	1.6708	2.6202	2.9362	2.9707

$1, 2, \ldots, 2J$, are the zeros of the Chebyshev polynomial $U_{2J}(x)$. For these values of β_r, one can always find positive weights c_r that solve (27). Actually, using the Gauss-Jacobi mechanical quadrature [16] for the abscissas x_r, one can prove that

$$c_r = \frac{16\sin\beta_r}{2J + 1} \sum_{n=0}^{J-1} \frac{J(J + 1) - n(n + 1)}{2n + 1} \times \sin[(2n + 1)\beta_r], \qquad 1 \leq r \leq 2J. \quad (28)$$

This solution has the nice feature of being simple and explicit, but it is not very economical. One can reduce the number of values β_r by taking x_r to be the $J + 1$ zeros of $P_{J+1}(x)$. In this case, c_r are the Christoffel numbers given by $c_r = \int_{-1}^{1} dx[P_{J+1}(x)/(x - x_r)P'_{J+1}(x)]^2 > 0$ [17].

The recipes above yield finite optimal POVM's for any value of N. In general, one can further reduce the number of outcomes. Ideally, one would be interested in finding the minimal POVM's; however, as far as we are aware, the solution is not known for arbitrary J and general groups [4]. Nevertheless, the minimal measurement for the first nontrivial case of two spins is not difficult to find. Consider the simplest normalized reference state that leads to an optimal POVM, i.e., $|B\rangle = \sqrt{3}/2|1,1\rangle + 1/2|0,0\rangle$. It is easy to verify that the four projectors $O_r = U(g_r)|B\rangle\langle B|U^\dagger(g_r)$, with

$$\alpha_r = \frac{2\pi(r - 1)}{3}, \qquad \gamma_r = \pi - \alpha_r, \qquad \cos\beta_r = -\frac{1}{3}, \qquad r \leq 3; \qquad (29)$$
$$\alpha_4 = 0, \qquad \gamma_4 = 0, \qquad \cos\beta_4 = 1,$$

satisfy the condition $\sum_{r=1}^{4} O_r = \mathbf{1}$. Since the Hilbert space has dimension four, the minimal number of outcomes for any measurement is also four. This POVM is therefore finite, minimal, and optimal. In fact, it is a von Neumann measurement, as $O_r O_s = \delta_{rs} O_r$.

We conclude that it is feasible to use quantum systems to encode the orientation of a reference frame. The optimal strategy involves the use of encoding states which are remarkably simple and have a clear physical interpretation. The average error of the transmission is seen to approach zero linearly in $1/N$. Finally, we give recipes for constructing finite optimal POVM's and present an example of a minimal one for the simple case $N = 2$.

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