Discrete Diffraction Managed Spatial Solitons

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Motivated by recent experimental observations of anomalous diffraction in linear waveguide array, we propose a novel model governing the propagation of an optical beam in a diffraction managed nonlinear waveguide array. This model supports discrete spatial solitons whose beamwidth and peak amplitude evolve periodically. A nonlocal integral equation governing the slow evolution of the soliton amplitude is derived and its stationary soliton solutions are obtained.

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Discrete spatial solitons in nonlinear media have attracted considerable attention in the scientific community for many years [1]. They have been demonstrated to exist in a wide range of physical systems such as atomic chains [2,3], molecular crystals [4], biophysical systems [5], electrical lattices [6], and recently in arrays of coupled nonlinear optical waveguides [7,8]. Such discrete excitations can form when discrete diffraction is balanced by nonlinearity. In an optical waveguide array this can be achieved by using an intense laser beam which locally changes the nonlinear refractive index of the waveguides via the Kerr effect and in turn decouples them from the remaining waveguides.

After almost a decade since their first theoretical prediction [9], discrete solitons in an optical waveguide array were experimentally observed [7]. When a low intensity beam is injected into one waveguide, the propagating field spreads over the adjacent waveguides, hence, experiencing discrete diffraction. However, at sufficiently high power, the beam self-trap (to form a localized state) in the center waveguides. Soon thereafter, many fascinating properties of discrete solitons were reported: for example, the experimental observation of linear and nonlinear Bloch oscillations in AlGaAs waveguides [10]; temperature tuned polymer waveguides [11], and in an array of curved optical waveguides [12]. Moreover, the dynamical behavior of discrete solitons was also experimentally realized [8]. Discrete solitons have unique properties that are absent in their bulk counterparts [13]. The most noticeable one is the possibility of designing the diffraction properties of a *linear* waveguide array to produce *anomalous* diffraction [14]. As a result, self-focusing and defocusing processes can be achieved in the same medium (structure) and wavelength that leads to the possibility of observing dark solitons in self-focusing Kerr media [15].

In this Letter, inspired by the recent experimental observations of *anomalous* diffraction in a *linear* waveguide array, we propose a novel model governing the propagation of an optical beam in a diffraction managed *nonlinear* waveguide array. Using a cascade of tilted segments of an array of waveguides with positive average diffraction, we develop a *nonlinear discrete model* that describes the evolution of an optical field in self-focusing nonlinear Kerr media in the presence of both normal and anomalous diffraction. Importantly, even though optical diffraction and chromatic dispersion originate from different physics (the former being geometric and the latter being media dependent), nevertheless, discrete diffraction spatial solitons and dispersion managed solitons share many properties which highlight the universality and diversity of solitons. A nonlocal integral equation governing the slow evolution of the beam amplitude is derived, and its stationary soliton solutions are obtained. Such averaged beams lose peak power during propagation, hence, since the overall power is preserved, they become wider; at the end point of the diffraction map, a full recovery of the initial beam peak power occurs. The competition between varying diffraction and self-focusing nonlinearity offers many new exciting physical possibilities.

We begin our analysis by considering an infinite array of weakly coupled optical waveguides with equal separation d. The equation which governs the evolution of the electric field E_n, according to nonlinear coupled mode theory [7,9,16], is given by

$$\frac{\partial E_n}{\partial z} = iC(E_{n+1} + E_{n-1}) + ik_w E_n + i\kappa|E_n|^2 E_n\,, \quad (1)$$

where C is the coupling constant between adjacent waveguides which is given by an overlap integral of the two modes of such waveguides; κ is a constant describing the nonlinear index change, z is the propagation distance, and k_w is the propagation constant of the waveguides. To facilitate understanding, we first recall basic properties of discrete diffraction of a *linear* array. When a cw mode of the form $E_n(z) = A\exp[i(k_z z - nk_x d)]$ is inserted into the linearized version of Eq. (1) it yields

$$k_z = k_w + 2C\cos(k_x d)\,. \quad (2)$$

In close analogy to the definition of dispersion, discrete diffraction is given by $k_z'' = -2Cd^2\cos(k_x d)$. As first pointed out in [14], an important feature of the last relation is that k_z'' can change sign depending on k_x. Indeed, the diffraction becomes positive in the range $\pi/2 < |k_x d| \le \pi$; hence, a light beam can experience *anomalous* diffraction. In practice, the sign and value of the diffraction can be controlled and manipulated by launching light at a particular angle or equivalently by tilting the waveguide array. This in turn allows the possibility of achieving a "self-defocusing" (with *positive* Kerr coefficient) regime which leads to the formation of discrete dark solitons [15]. Similar to [14], we use a cascade of different segments of waveguide, each piece being tilted by an angle zero, and γ, respectively. The actual physical angle γ (the waveguide tilt angle) is related to the wave number k_x by the relation [14] $\sin\gamma = k_x/k$, where $k = 2\pi n_0/\lambda_0$ ($\lambda_0 = 1.53\ \mu$m is the central wavelength in vacuum and $n_0 = 3.3$ is the linear refractive index). In this way, we generate a waveguide array with alternating diffraction. To model such a system, we define $E_n = \sqrt{P_*}\,\phi_n e^{i(k_w+2C)z}$ and $z' = z/z_*$, with P_* and z_* being the characteristic power and nonlinear length scale, respectively. Substituting these quantities into Eq. (1) yields the following (dropping the prime) diffraction-managed discrete nonlinear Schrödinger (DM-DNLS) equation:

$$\frac{d\phi_n}{dz} = i\,\frac{D(z/z_w)}{2h^2}(\phi_{n+1} + \phi_{n-1} - 2\phi_n) + i|\phi_n|^2\phi_n, \tag{3}$$

with $z_* = 1/(\kappa P_*)$ and $z_* C\cos(k_x d) = D(z/z_w)/(2h^2)$, where $D(z/z_w)$ is a piecewise constant periodic function that measures the local value of diffraction. Here, $z_w \equiv 2L/z_*$, with L being the actual length of each waveguide segment [see Fig. 1(a) for schematic representation]. Equation (3) describes the dynamical evolution of a laser beam in a Kerr medium with varying diffraction. When the "effective" nonlinearity balances the average diffraction, then bright discrete solitons can form.

The coupling constants that correspond to the experimental data reported in [14] (for 2.5 μm waveguide separation and width) are found to be $C = 2.27\ \mathrm{mm}^{-1}$ [17], $\kappa = 3.6\ \mathrm{m}^{-1}\,\mathrm{W}^{-1}$. For typical power $P_* \approx 300$ W and waveguide length $L \approx 100\ \mu$m, we find $z_* \approx 1$ mm and $z_w \approx 0.2$. Hence, it is natural to construct an asymptotic theory based on small z_w. It is in this parameter regime that diffraction managed spatial solitons can be experimentally observed. To this end, we consider the case in which the diffraction takes the form

$$D(z/z_w) = \delta_a + \frac{1}{z_w}\Delta(z/z_w), \tag{4}$$

where δ_a is the average diffraction (taken to be positive) and $\Delta(z/z_w)$ is a periodic function. Since in this case Eq. (3) contains both slowly and rapidly varying terms, we introduce new fast and slow scales as $\zeta = z/z_w$ and $Z = z$, respectively, and expand ϕ_n in powers of z_w:

$$\phi_n = \phi_n^{(0)}(\zeta, Z) + z_w\phi_n^{(1)}(\zeta, Z) + O(z_w^2). \tag{5}$$

Substituting Eqs. (5) and (4) into Eq. (3), we find that the leading order in $1/z_w$ and the order 1 equations are, respectively, given by

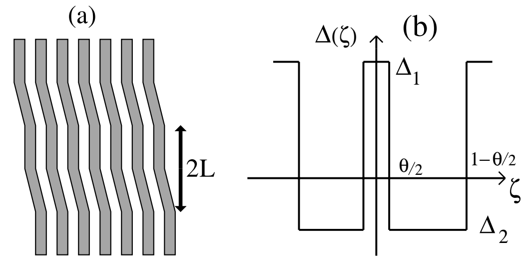

FIG. 1. Schematic presentation of the waveguide array (a) and of the diffraction map (b).

$$J(\phi_n^{(0)}) = 0, \qquad J(\phi_n^{(1)}) = -\mathcal{F}_n, \tag{6}$$

where

$$J(A_n) \equiv i\,\frac{\partial A_n}{\partial\zeta} + \frac{\Delta(\zeta)}{2h^2}(A_{n+1} + A_{n-1} - 2A_n),$$

$$\mathcal{F}_n = i\,\frac{\partial\phi_n^{(0)}}{\partial Z} + \frac{\delta_a}{2h^2}(\phi_{n+1}^{(0)} + \phi_{n-1}^{(0)} - 2\phi_n^{(0)}) + |\phi_n^{(0)}|^2\phi_n^{(0)}.$$

To solve at order $1/z_w$ [see Eq. (6)], we introduce the discrete Fourier transform,

$$\hat\phi_0(q,\zeta,Z) = \sum_{n=-\infty}^{+\infty}\phi_n^{(0)}(\zeta,Z)e^{-iqnh},$$
$$\phi_n^{(0)}(\zeta,Z) = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h}\hat\phi_0(q,\zeta,Z)e^{iqnh}\,dq. \tag{7}$$

The solution is therefore given in the Fourier representation by

$$\hat\phi_0(q,\zeta,Z) = \hat\psi(Z,q)\exp[-i\Omega(q)C(\zeta)], \tag{8}$$

with $\Omega(q) = [1-\cos(qh)]/h^2$ and $C(\zeta) = \int_0^\zeta \Delta(\zeta')\,d\zeta'$.

The amplitude $\hat\psi(Z,q)$ is an arbitrary function whose dynamical evolution will be determined by a secularity condition associated with Eq. (6). In other words, the condition of the orthogonality of $\mathcal{F}_n$ to all eigenfunctions Φ_n of the adjoint linear problem which, when written in the Fourier domain, takes the form

$$\int_0^1 d\zeta\,\hat{\mathcal{F}}(q,\zeta,Z)\hat\Phi(q,\zeta,Z) = 0, \tag{9}$$

where $\hat{\mathcal{F}}$, $\hat\Phi$ are the Fourier transform of $\mathcal{F}_n$, Φ_n, respectively. Here, $\hat J^\dagger\hat\Phi = 0$ with $\hat J^\dagger$ being the adjoint operator to $\hat J \equiv i d/d\zeta - \Delta(\zeta)\Omega(q)$. Substituting (8) into $\hat{\mathcal{F}}$ and performing the integration in condition (9) yields the following nonlinear evolution equation for $\hat\psi(Z,q)$:

$$i\,\frac{d\hat\psi(Z,q)}{dZ} = \delta_a\Omega(q)\hat\psi(Z,q) - \mathcal{R}[\hat\psi(Z,q)],$$

$$\mathcal{R} \equiv \int d\mathbf{q}\,\mathcal{K}(q,q_1,q_2)\hat\psi(q_1)\hat\psi(q_2) \times \hat\psi^*(q_1+q_2-q), \tag{10}$$

where $d\mathbf{q} \equiv dq_1dq_2$ and the kernel $\mathcal{K}$ is defined by

$$\mathcal{K}(q, q_1, q_2) = \frac{h^2}{4\pi^2} \int_0^1 d\zeta \exp[iC(\zeta)\chi(q, q_1, q_2)],$$

$$\chi = \frac{4}{h^2} \cos\left(\frac{h(q_1 + q_2)}{2}\right) \prod_{j=1}^{2} \sin\left(\frac{h(q_j - q)}{2}\right).$$

Equation (10) governs the evolution in Fourier space of an optical beam in a coupled nonlinear waveguide array in the regime of strong diffraction. In the special case of the two-step diffraction map shown in Fig. 1(b), i.e., when two waveguide segments are tilted by angle zero and γ alternatively, we have $\Delta(\zeta) = \Delta_1$ for $0 \leq |\zeta| < \theta/2$ and Δ_2 in the region $\theta/2 < |\zeta| < 1/2$, where θ is the fraction of the map with diffraction Δ_1. In this case, the kernel $\mathcal{K}$ takes the simple form $\mathcal{K} = h^2 \sin(s\chi)/(4\pi^2 s\chi)$ with $s = [\theta\Delta_1 - (1 - \theta)\Delta_2]/4$. Importantly, these parameters can be related to the experiments reported in [14]. To achieve a waveguide configuration with alternate diffraction, we use two values of $k_x d = 0$ and $2\pi/3$ ($h = 1$) which corresponds to waveguide tilt angles $\gamma = 0$ and 3.43°. In this case we find, for $\theta = 0.5$, $\Delta_1 = -\Delta_2 = 0.681$, $\delta_a = 1.135$, and $s = 0.17$. Different sets of parameters with a smaller angle γ are also realizable. Next, we look for a stationary solution for Eq. (10) (for the particular kernel given above) in the form $\hat{\psi}(Z, q) = \hat{\psi}_s(q) \exp(i\omega_s Z)$. Inserting this ansatz into (10) leads to

$$\hat{\psi}_s(q) = \frac{1}{\delta_a \Omega(q) + \omega_s} \mathcal{R}[\hat{\psi}_s(q)] \equiv \mathcal{M}[\hat{\psi}_s(q)], \quad (11)$$

which implies that the mode $\hat{\psi}_s(q)$ is a fixed point of the nonlinear functional $\mathcal{M}$. To numerically find the fixed point, we employ a modified Neumann iteration scheme [18–20] and write Eq. (11) in the form

$$\hat{\psi}_s^{(m+1)}(q) = \left(\frac{\alpha(\hat{\psi}_s^{(m)})}{\beta(\hat{\psi}_s^{(m)})}\right)^{3/2} \mathcal{M}[\hat{\psi}_s^{(m)}(q)], \qquad m \geq 0,$$

$$\alpha = \int |\hat{\psi}_s^{(m)}(q)|^2 \, dq\,;$$

$$\beta = \int \hat{\psi}_s^{(m)}(q) \mathcal{M}[\hat{\psi}_s^{(m)}(q)] \, dq\,.$$

The factors α and β are introduced to stabilize an otherwise divergent Neumann iteration scheme. This method is used to find stationary soliton solutions to the integral equation (10) which in turn provides an asymptotic description of the diffraction-managed DNLS Eq. (3). It should be also noted that we can obtain periodic diffraction-compensated soliton solutions directly from Eq. (3). The technique is similar to that originally proposed for finding periodic dispersion-managed solitons in communications problems [21,22]. The averaging procedure does not require that the map period, z_w, be small. To implement the method, initially we start with a guess, for instance $\phi_n^{(0)} = \text{sech}(nh)$ with $\mathcal{E}_0 = \sum_{n=-\infty}^{\infty} \text{sech}^2(nh)$. Over one period, this initial ansatz will evolve to $\phi_n^{(0)'}$ which in general will have a chirp [23]. We then define an average: $\phi_n^{(0)''} = (\phi_n^{(0)} + \phi_n^{(0)'} e^{-i\Theta_n})/2$, where $\phi_n^{(0)'} = |\phi_n^{(0)'}| \exp(i\Theta_n)$ which has power $\mathcal{E}_0''$. Then $\phi_n^{(1)} = \phi_n^{(0)''}\sqrt{\mathcal{E}_0/\mathcal{E}_0''}$ is the new guess and, in general, the mth iteration takes the form

$$\phi_n^{(m+1)} = \sqrt{\mathcal{E}_0/\mathcal{E}_m''}\, \phi_n^{(m)''}, \qquad \mathcal{E}_m'' = \sum_{n=-\infty}^{\infty} |\phi_n^{(m)''}|^2. \quad (12)$$

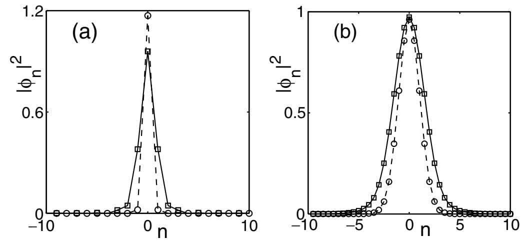

FIG. 2. Mode profile in physical space obtained from Eq. (11) (solid line) and from the average method (dashed line). Parameters are as follows: (a) $\omega_s = 1$, $h = 1$, $s = 0.17$, $\Delta_1 = -\Delta_2 = 0.681$, $\delta_a = 1.135$, and $z_w = 0.2$; (b) $\omega_s = 1$, $h = 0.5$, $s = 1$, $\Delta_1 = -\Delta_2 = 4$, $\delta_a = 1$, and $z_w = 0.2$.

In Fig. 2 the mode profiles associated with a stationary solution are depicted for two typical parameter values. The profiles are obtained by using both the integral equation approach as well as the averaged method. The evolution of these discrete diffraction managed solitons are illustrated in Figs. 3 and 4 for the same two sets of parameter values as in Fig. 2, respectively. We note that initially the beam has zero chirp [23]. During propagation, a chirp develops and the peak amplitude of the beam begins to decrease and, as a result, the beam becomes wider (due to conservation of power). A full recovery of the soliton's initial amplitude and width is achieved at the end of the map period. This breathing behavior is shown in Figs. 3 and 4 for both strongly and moderately confined beams, respectively.

To establish the relation between the two approaches and to highlight the periodic nature of these new solitons, we calculate the nonlinear chirp by both the integral equation approach and the averaging method. It is clear that for small values of the map period, z_w, the asymptotic analysis is in good agreement with the averaging method, as shown in Fig. 5. We also mention briefly that the method of analysis associated with Eq. (3) can be modified to account for situations where the average diffraction is small,

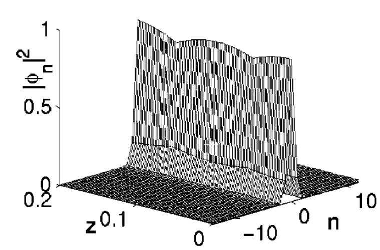

FIG. 3. Beam propagation over one period using Eq. (8) as the initial condition obtained by a direct numerical simulation of Eq. (3). Parameters are the same as used in Fig. 2(a).

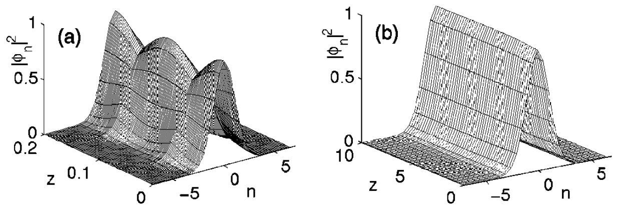

FIG. 4. Beam propagation over one period (a) and stationary evolution (b) obtained by a direct numerical simulation of Eq. (3) evaluated at end map period. Parameters are the same as used in Fig. 2(b).

i.e., $\delta_a \ll 1$. In such a situation we write $\delta_a = \epsilon D_a$, $D = \epsilon D_a + \Delta(z)$, $z = \xi/\epsilon$, and $\phi_n = \sqrt{\epsilon}\,\Psi_n$. Then it is found that Ψ_n satisfies

$$\frac{d\Psi_n}{d\xi} = i\,\frac{\mathcal{D}(\xi/\epsilon)}{2h^2}(\Psi_{n+1} + \Psi_{n-1} - 2\Psi_n) + i|\Psi_n|^2\Psi_n\,, \qquad (13)$$

where $\mathcal{D}(\xi/\epsilon) = D_a + \frac{1}{\epsilon}\Delta(\xi/\epsilon)$. The model (13) is valid in parameter regimes which applies to a physical situation where the average diffraction is small.

In conclusion, we have developed a new discrete equation governing the evolution of an optical beam in a waveguide array with varying diffraction. This equation has a novel type of discrete spatial soliton solution which breathes under propagation and, as a result, gains a nonlinear chirp. A full recovery of the soliton initial power is achieved at the end of each diffraction map. This opens the possibility of fabricating a customized waveguide array which admits specialized diffraction-managed spatially confined solitary waves. A nonlocal integral equation governing the slow evolution of the soliton amplitude is derived and its stationary solutions are obtained.

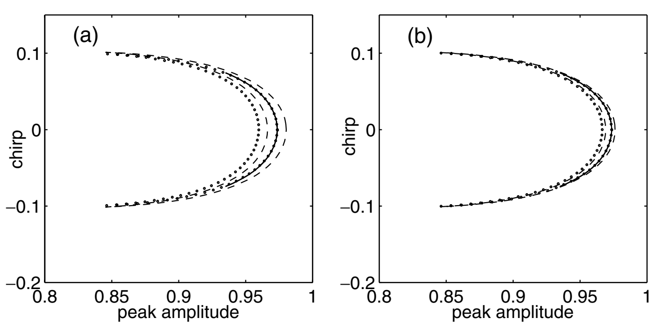

FIG. 5. Periodic evolution of the beam chirp versus maximum peak power. Solid line represents the leading order approximation, i.e., Eq. (8); dashed line represents numerical solution of the DM-DNLS, and dotted line depicts the evolution of the leading order solution under Eq. (3). Parameters are the same as used in Fig. 2(b) with $z_w = 0.2$ (a) and 0.1 (b).

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[23] In close analogy to the definition of a chirp in dispersion managed solitons, here, the chirp of a discrete beam $\phi_n(z) = |\phi_n(z)|\exp[i\Theta_n(z)]$ is defined to be the coefficient of n^2 in the expansion of $\Theta_n(z)$ in a "Taylor series" around $n = 0$. In other words, if $\Theta_n(z) \approx c_0(z) + c_1(z)n + c_2(z)n^2 + \ldots$, then the chirp is given by $c_2(z)$. This definition is valid for moderately localized solitons and breaks down for strongly localized beams.