Proposal for Determining the Total Masses of Eccentric Binaries Using Signature of Periastron Advance in Gravitational Waves

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We propose a new method for determining total masses of low frequency eccentric binaries (such as neutron star binaries with orbital frequency $f \gtrsim 10^{-3}$ Hz) from their gravitational waves. In this method we use the frequency shift caused by periastron advance, and it works even at low frequency band where the chirp signal due to radiation reaction is difficult to be measured. It is shown that the total masses of several Galactic neutron star binaries might be measured accurately (within a few percent error) by LISA with an operation period of ~10 yr.

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I. Introduction.—The Galactic binaries such as close white dwarf binaries (CWDBs) or neutron star binaries (NBs) are promising targets of the Laser Interferometer Space Antenna (LISA) [1]. One of the important aims of gravitational-wave astronomy is to extract out information of sources that emit gravitational radiation. In the case of a Galactic binary we want to know masses of two stars, orbital parameters (semimajor axis, eccentricity, inclination), distance to the binary, and so on. At the final in-spiral phase (e.g., the last three minutes of NBs) we can, in principle, determine various parameters by fitting time evolution of the wave signal with post-Newtonian expansion [2]. But at the LISA band many compact binaries evolve very slowly and the situation is largely different.

It is observationally known that NBs [and some neutron star–white dwarf (NS-WD) binaries] with orbital period $\lesssim$1 day have large eccentricities [3]. This is explained by the "kick" effect and instantaneous mass loss at birth of a neutron star, in contrast to CWDBs whose orbits are circularized by strong tidal interaction during mass transfer phases. As the eccentricity e decreases with an increase of the orbital frequency due to radiation reaction [4], its effect is supposed to be negligible for most high frequency sources that will be searched by ground-based detectors (TAMA300, GEO600, LIGO, and VIRGO). But the eccentricity of NBs can be $e \sim 0.1$ at frequency $f \sim 10^{-3}$ Hz as expected from PSR B1913 + 16 [5], and it might cause observable effects for LISA.

Frequency series of a gravitational wave from an eccentric binary is affected by the periastron advance [6] (see Ref. [5] for radio observation of binary pulsars). Using this fact we might obtain information of binaries even at a low frequency band where a chirp signal would be difficult to be measured. In this Letter we propose a new method for determining binary parameters (mainly total mass) and investigate its prospect for studying Galactic NBs with LISA.

II. Gravitational wave from elliptical orbit.—Let us study gravitational waves from elliptical orbits following Ref. [6] (see also [7]). With a quadrupole formula of gravitational radiation [8] two polarization modes in transverse-traceless gauge are written as follows:

$$h_\times = -h_0 \cos\Theta \sum_n \left[\frac{S_n - C_n}{2} \sin(2\pi f_n t + 2\Phi) + \frac{S_n + C_n}{2} \sin(2\pi f_n t - 2\Phi) \right], \tag{1}$$

$$h_+ = -\frac{1}{2} h_0 \sum_n \Big\{ \sin^2\Theta A_n \cos(2\pi f_n t) + (1 + \cos^2\Theta) \times \left[\frac{S_n - C_n}{2} \cos(2\pi f_n t + 2\Phi) + \frac{S_n + C_n}{2} \cos(2\pi f_n t - 2\Phi) \right] \Big\}, \tag{2}$$

where Θ represents the direction of the orbital angular momentum (inclination) and Φ represents the direction of the periastron in the orbital plane (we put $\Phi = 0$ when the periastron is on the plane determined by two vectors: the orbital angular momentum vector and the direction vector to the observer [6]). The amplitude h_0 is given as $h_0 = 4(2\pi)^{-2/3} G^{5/3} c^{-4} m_{\rm chirp}^{5/3} r^{-1} (P_b)^{-2/3}$ ($m_{\rm chirp}$: chirp mass, r: distance to the binary, P_b: orbital period from periastron to periastron). We have defined the frequency series $f_n \equiv P_b^{-1} n$. The angle Φ changes due to the periastron advance and we can denote it as $2\Phi = \delta f \times t$ by adjusting the origin of the time coordinate t. Thus the periastron advance causes frequency shift δf that is given in terms of the total mass $m_{\rm total}$ and the eccentricity e as $\delta f = 6(2\pi)^{2/3} (P_b)^{-5/3} G^{2/3} c^{-2} m_{\rm total}^{2/3} (1 - e^2)^{-1}$ (twice the frequency for the periastron advance [8]). The frequency f_n now splits into a triplet ($f_n - \delta f, f_n, f_n + \delta f$). The effect of the angle 2Φ is related to rotation of a coordinate system and appears in the same form for every n mode. This is an essential point. The simple replacement $2\Phi \rightarrow 2\delta f \times t$ into expressions (1) and (2) corresponds to an

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approximation that neglects terms of $O(P_b \delta f) \ll 1$ [6]. The coefficients S_n, C_n, and A_n are given by the nth Bessel function $J_n(x)$ and its derivative $J_n'(x) \equiv \partial_x J_n(x)$ as

$$A_n = J_n(ne), \tag{3}$$

$$S_n = -\frac{2(1-e^2)^{1/2}}{e} J_n'(ne) + \frac{2n(1-e^2)^{3/2}}{e^2} J_n(ne),$$

$$C_n = -\frac{2-e^2}{e^2} J_n(ne) + \frac{2(1-e^2)}{e} J_n'(ne). \tag{4}$$

We can expand these coefficients around $e = 0$ (circular orbit) and find that only $n \leq 3$ modes have terms $O(e^0)$ or $O(e^1)$. They are given as $S_1 = -\frac{3}{4}e + O(e^3)$, $C_1 = -\frac{3}{4}e + O(e^3)$, $A_1 = \frac{1}{2}e + O(e^3)$ for the $n = 1$ mode, $S_2 = 1 - \frac{5}{2}e^2 + O(e^4)$, $C_2 = 1 - \frac{5}{2}e^2 + O(e^4)$, $A_2 = \frac{1}{2}e^2 + O(e^4)$ for the $n = 2$ mode, and $S_3 = -\frac{9}{4}e + O(e^3)$, $C_3 = -\frac{9}{4}e + O(e^3)$, $A_3 = O(e^3)$ for the $n = 3$ mode. Our basic strategy is to compare the $O(e^0)$ term of the $n = 2$ mode and the $O(e^1)$ term of the $n = 3$ mode (as we see below, the eccentricity relevant for our analysis is expected to be small $e \lesssim 0.1$). From these two frequencies $f_2 - \delta f$ and $f_3 - \delta f$ we obtain

$$3(f_2 - \delta f)/2 - (f_3 - \delta f) = -\delta f/2, \tag{5}$$

and thus the total mass $m_{\rm total}$ is estimated for a binary with small eccentricity e (note also we might determine the eccentricity e by comparing amplitudes of two waves). The amplitude of the $n = 1$ mode is smaller than the $n = 3$ mode, and furthermore the binary confusion noise would be larger at lower frequency [1]. Therefore we investigate the prospect of our method by studying the detectability of the $n = 3$ mode instead of the $n = 1$ mode.

III. Detectability.—The frequency difference δf due to periastron advance is expressed as [8]

$$\delta f = \frac{1.2 \times 10^{-7}}{1-e^2} \left(\frac{m_{\rm total}}{2.8 M_\odot}\right)^{2/3} \left(\frac{f_3}{3 \times 10^{-3}\ {\rm Hz}}\right)^{5/3} {\rm Hz}, \tag{6}$$

and the accumulated frequency difference (chirp signal) due to the gravitational radiation reaction within the observational period $T_{\rm obs}$ is given as [4]

$$(\dot f)_{GW} T_{\rm obs} = \frac{3.0 \times 10^{-8}}{(1-e^2)^{7/2}} \left(\frac{m_{\rm chirp}}{1.2 M_\odot}\right)^{5/3} \times \left(\frac{f_2}{2 \times 10^{-3}\ {\rm Hz}}\right)^{11/3} \times \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right) \left(\frac{T_{\rm obs}}{10\ {\rm yr}}\right) {\rm Hz}. \tag{7}$$

Thus at lower frequency the difference due to the periastron advance can take a larger value. The estimation error (resolution) Δf for wave frequency f in the matched filtering analysis is given by the Fisher information matrix of fitting parameters. For the set of three unknown parameters ($f, \dot f$, initial phase) of a quasimonochromatic wave ($\dot f T_{\rm obs} \ll f$) we obtain [9]

$$\Delta f = 4\sqrt{3}\, \pi^{-1} T_{\rm obs}^{-1} {\rm SNR}^{-1} = 6.6 \times 10^{-10} \left(\frac{T_{\rm obs}}{10\ {\rm yr}}\right)^{-1} \left(\frac{\rm SNR}{10}\right)^{-1} {\rm Hz}, \tag{8}$$

where SNR is the signal to noise ratio of the wave. For the secular frequency modulation $\dot f$ we have the estimation error $\Delta \dot f = 3\sqrt{5}\, \pi^{-1} T_{\rm obs}^{-2} {\rm SNR}^{-1}$. (When we include the direction and orientation of the source in fitting parameters, the estimation errors might be somewhat larger than our evaluation. We should also notice that these angular parameters are fixed well by the $f = 2$ mode and we might not need to fit them for determining the frequency of the $f = 3$ mode.) From Eqs. (5), (6), and (8) the frequency shift δf would be resolved:

$$\frac{\Delta f}{\delta f} \sim 0.011(1-e^2) \left(\frac{m_{\rm total}}{2.8 M_\odot}\right)^{-2/3} \left(\frac{f_3}{3 \times 10^{-3}\ {\rm Hz}}\right)^{-5/3} \times \left(\frac{\rm SNR}{10}\right)^{-1} \left(\frac{T_{\rm obs}}{10\ {\rm yr}}\right)^{-1}. \tag{9}$$

Here (and hereafter) we consider the contribution of the estimation error for δf only from the $n = 3$ mode, as the SNR of the $n = 2$ mode would be larger. In the following we estimate the number of Galactic NBs whose $n = 3$ mode is detected with SNR > 10.

First we discuss time evolution of the eccentricity parameter e. Using a quadrupole formula for gravitational radiation the orbital semimajor radius a is related to the eccentricity as $a/a_i = (1 - e_i^2)/(1 - e^2)(e/e_i)^{12/19}[(1 + 121e^2/304)/(1 + 121e_i^2/304)]^{870/2299}$, where a_i and e_i are their initial values [4]. At $a/a_i \ll 1$ we have a useful approximation $e \sim [a/a_i(1 - e_i^2)]^{19/12} e_i = (f/f_i)^{-19/18}(1 - e_i^2)^{-19/12} e_i$, where Kepler law is used. For a given orbital period the eccentricity should have some distribution function $F(e; P_b)$ [10], but it is poorly known at present, as only several NBs have been detected so far [3]. Here we use the binary pulsar PSR B1913 + 16 as a reference. This system has orbital period $P_b = 2.8 \times 10^4$ sec and eccentricity $e = 0.62$ [5]. From the above approximation its eccentricity evolves as

$$e \simeq 0.13(f_3/0.001\ {\rm Hz})^{-19/18}. \tag{10}$$

Hereafter we use this e-f_3 relation in our order estimation.

Next we study the frequency and spatial distribution of the Galactic NBs. Assuming that the former is in the steady state, we can evaluate the distribution function dN/df using the coalescence rate $R_{\rm NS}$ of Galactic NBs as

$$\frac{dN}{df} = R_{\rm NS} \left(\frac{df}{dt}\right)^{-1} \sim 3.8 \times 10^4 \left(\frac{R_{\rm NS}}{10^{-5}\ {\rm yr}^{-1}}\right) \left(\frac{f_2}{10^{-3}\ {\rm Hz}}\right)^{-11/3} {\rm Hz}^{-1}. \tag{11}$$

Kalogera *et al.* [11] recently estimated the event rate as $R_{\rm NS} \simeq 10^{-6} - 5 \times 10^{-4}\ {\rm yr}^{-1}$ (see also [12]). For a spatial distribution of Galactic NBs we use the standard exponential disk model $\rho(R,z) = \rho_0 \exp(-R/R_0) \times \exp(-|z|/z_0)$, where (R,z) is the Galactic cylindrical coordinate [11,13]. We fix the radial scale length $R_0 = 3.5$ kpc and the disk scale height $z_0 = 500$ pc and assume that the solar system exists at the position $R = 8.5$ kpc and $|z| = 30$ pc.

We have a relation $h_3(f_3) \simeq -9eh_2(2f_3/3)/4$ between the $n = 2$ mode and the $n = 3$ mode [see Eqs. (1) and (2)]. The amplitude h_3 of the $n = 3$ mode is given as

$$h_3 \simeq 4.4 \times 10^{-21}\left(\frac{m_{\rm chirp}}{1.2M_\odot}\right)^{5/3}\left(\frac{f_3}{3\times 10^{-3}\ {\rm Hz}}\right)^{2/3} \times \left(\frac{100\ {\rm pc}}{r}\right)\left(\frac{e}{0.1}\right), \qquad (12)$$

where we have taken an angular average with respect to orientation of sources [14]. From Eqs. (10) and (12) we can estimate the distance to a binary whose $n = 3$ mode is detected with a given SNR,

$$r = 1800\left(\frac{f_3}{3\times 10^{-3}\ {\rm Hz}}\right)^{-7/18} \times \left(\frac{h_{\rm rms}(f_3, T_{\rm obs})}{10^{-23}}\right)^{-1}\left(\frac{10}{\rm SNR}\right)\ {\rm pc}, \qquad (13)$$

where $h_{\rm rms}(f_3, T_{\rm obs})$ is the noise spectrum. We use the angular averaged sensitivity (effectively a factor of $\sqrt{5}$ degradation [1,14]) to take rotation of LISA into account and adopt the noise spectrum given in Ref. [15]. When the effective frequency bin $T_{\rm obs}^{-1}$ is occupied by more than one Galactic CWDB (more precisely, the number of fitting parameters is larger than that of the data), their confusion noise becomes important and the sensitivity $h_{\rm rms}(f, T_{\rm obs})$ would be significantly worse at lower frequency. Then binaries only very close to the solar system would be resolved. This transition occurs at frequency $f_t \simeq 1.6 \times 10^{-3}(T/10\ {\rm yr})^{-3/11}$ Hz for the abundance of Galactic CWDBs estimated in Ref. [15]. As we see below, the prospect of our method depends strongly on the transition frequency and the position of f_t is very important. In an extreme model that the Galactic massive compact halo objects are WDs, the frequency f_t can be close to $\sim 10^{-2}$ Hz [16] (see also [17]). If this is true, our method would be severely limited. Spatial filters that depend on the angular position of sources might work effectively in some cases [18]. At $f \gtrsim f_t$ the noise spectrum $h_{\rm rms}(f, T_{\rm obs})$ is mainly determined by the detector's intrinsic noise and the confusion noise by extra-Galactic CWDBs that depends on cosmological evolution of binary systems [1,13,19]. At this frequency region the total noise spectrum behaves simply as $h_{\rm rms}(f, T_{\rm obs}) \propto T_{\rm obs}^{-1/2}$. In the present analysis we count NBs only with a frequency $f_2 = 2f_3/3 > f_t$.

Now we can estimate the number of NBs whose $n = 3$ mode is detectable by LISA with SNR > 10. In Fig. 1 we show the results for observational period $T_{\rm obs} = 1$ yr and $T_{\rm obs} = 10$ yr. The total number (integrated in frequency space) becomes $0.11(R_{\rm NS}/10^{-5}\ {\rm yr})$ for 1 yr observation and $3.7(R_{\rm NS}/10^{-5}\ {\rm yr})$ for 10 yr. Thus our method would be effective for a long (but realistic) observational period [1]. For these NBs ($T_{\rm obs} = 10$ yr, SNR ≤ 10 and $f_2 > f_t$) the total masses $m_{\rm total}$ would be resolved within a few percent accuracy [see Eq. (9)]. In Fig. 1 we also plot all the Galactic NBs (dashed line). Note that our method works well for Galactic NBs with $f > f_t$ in the case of $T_{\rm obs} = 10$ yr. The estimation error for the overall amplitude of wave becomes $\sim 0.15({\rm SNR}/10)^{-1}$ in the LISA band (see Table 4.5 of [1]). Thus the eccentricity parameter e for a detected NB (with SNR > 10 for the $n = 3$ mode) would be measured within $\sim 15\%$ accuracy. It has been observationally clarified that the mass of a neutron star is $\sim 1.4M_\odot$ [20] (that of an ordinal white dwarf is $\lesssim 1.2M_\odot$). If most eccentric binaries with $m_{\rm total} \lesssim 2.8M_\odot$ and $f \gtrsim 10^{-3}$ Hz are either NS + NS or NS-WD binary [13], we can obtain further information of such binaries detected by our method in the following manner. One of the two stars would be a neutron star with mass $\sim 1.4M_\odot$, and we can estimate the chirp mass of the system. Then the distance r to the binary is obtained as proposed by Schutz [21].

Using a similar method based on the expression $\Delta\dot{f}$ given just after Eq. (8), we have also estimated the number of Galactic NBs whose chirp signal $\dot{f}_{GW}$ for the $n = 2$ mode due to gravitational radiation is measured within 5% accuracy [$5 \times 3/5 \sim 3\%$ for the chirp mass

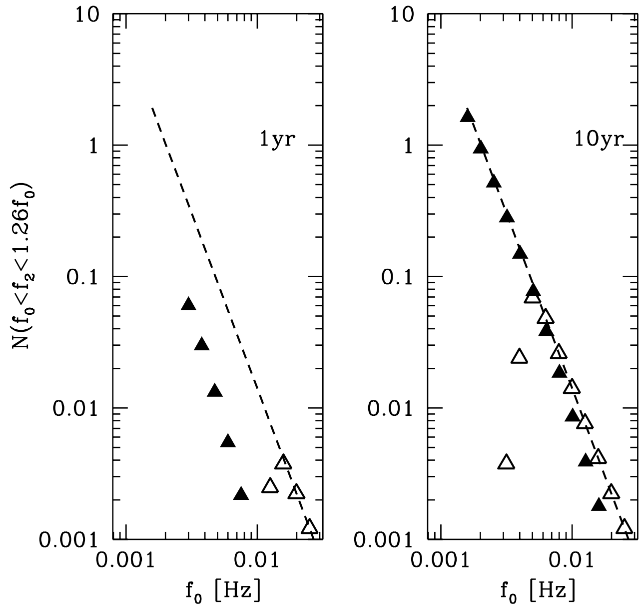

FIG. 1. Various distributions of Galactic NBs within frequency bins $f_0 \le f_2 < 1.26 f_0$. The dashed line represents all Galactic NBs. The filled triangles are the number of NBs whose gravitational wave of the $n = 3$ mode are detected with SNR ≥ 10, and open triangles are those whose chirp signals $(\dot{f})_{GW}$ due to radiation reaction are measured within 5% accuracy. We take the coalescence rate of Galactic NBs at $R_{\rm NS} = 10^{-5}\ {\rm yr}^{-1}$.

Eq. (7)] [9]. As shown in Fig. 1, all Galactic NBs with $f_2 \gtrsim 4 \times 10^{-3}$ Hz satisfy this observational criteria for $T_{\rm obs} = 10$ yr. There might be $\sim 1(R_{\rm NS}/10^{-5}\ {\rm yr}^{-1})$ NBs whose individual masses can be determined accurately from observed $m_{\rm total}$ and $m_{\rm chirp}$.

One might wonder whether two gravitational waves from different binaries (mainly CWDBs) are confused as $n = 2$ and $n = 3$ modes of the same eccentric source. The frequency distribution of Galactic close white dwarf binaries is estimated as $dN/df \sim 1.4 \times 10^9 (f/10^{-3}\ {\rm Hz})^{-11/3}\ {\rm Hz}^{-1}$ [15]. We can specify the direction of a monochromatic source with an estimation error $\lesssim 10^{-2}$ sr for frequency $f \gtrsim 2 \times 10^{-3}$ Hz (in the case of a signal to noise ratio: 10, Table 4.5 of Ref. [1]) using annual modulation of the gravitational wave due to the motion of LISA [1,22]. The mean frequency interval $(\delta f)_{\rm int}$ for binaries within a box of $\sim 10^{-2}$ sr is given as $(\delta f)_{\rm int} \sim (10^{-2}/4\pi)^{-1}(dN/df)^{-1} \sim 6 \times 10^{-5}(f/4 \times 10^{-3}\ {\rm Hz})^{11/3}$ Hz. The typical frequency difference due to periastron advance δf [Eq. (6)] is much smaller than this interval and coincidence of misidentification would be $(\delta f)/(\delta f)_{\rm int} \sim 2 \times 10^{-3}(f_3/3 \times 10^{-3}\ {\rm Hz})^{-2}$. The total number of Galactic CWDBs within frequency $2 \times 10^{-3}\ {\rm Hz} < f_2 < 4 \times 10^{-3}$ Hz is estimated as $\sim 4 \times 10^4$. Thus number of misidentified CWDB pairs would be at most ~ 80. The estimation error of LISA for the direction of the orbital angular momentum is $\sim 10^{-1}$ sr (SNR = 10, Table 4.5 of Ref. [1]). Thus there would be only $\sim 80 \times 10^{-1}/4\pi \sim 1$ misidentified CWDBs. It would also be possible to distinguish NBs using the estimated total mass itself, as most CWDBs have total masses smaller than $2.4 M_\odot$ [13].

Let us summarize this Letter. We have proposed a new method to determine the total masses of low frequency eccentric binaries from their gravitational waves. We use effects of periastron advance on the frequency space of the gravitational wave. Our method is effective for Galactic NBs ($f \gtrsim 2 \times 10^{-3}$ Hz) with a long term (~ 10 yr) operation of LISA.

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