# Coherence-Preserving Quantum Bits

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Real quantum systems couple to their environment and lose their intrinsic quantum nature through the process known as decoherence. Here we present a method for minimizing decoherence by making it energetically unfavorable. We present a Hamiltonian made up solely of two-body interactions between four two-level systems (qubits) which has a 2-fold degenerate ground state. This degenerate ground state has the property that any decoherence process acting on an individual physical qubit *must supply energy from the bath to the system.* Quantum information can be encoded into the degeneracy of the ground state and such coherence-preserving qubits will then be robust to local decoherence at low bath temperatures. We show how this quantum information can be universally manipulated and indicate how this approach may be applied to a quantum dot quantum computer.

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One of the most severe experimental difficulties in quantum information processing is the fragile nature of quantum information. Every real quantum system is an open system which readily couples to its environment. This coupling causes the quantum information in the system to become entangled with its environment, which in turn results in the system information losing its intrinsic quantum nature. This process is known as decoherence. Circumvention of this *decoherence problem* has been shown to be theoretically possible with the development of the theory of fault-tolerant quantum error correction [1]. The set of requirements to reach the threshold for such fault-tolerant quantum computation is, however, extremely daunting. In this Letter we present a quantum informatics method for suppressing the detrimental effects of decoherence, while at the same time allowing for robust manipulation of the quantum information, in the hope that this method will aid in breeching the threshold for robust quantum computation [2].

In the absence of coupling between a system and its environment, the system and environment have separate temporal evolutions determined by their individual energy spectra. When a small interaction (relative to these energy scales) is switched on between the two, the resulting evolution is dominated by pathways that conserve the energy of the unperturbed system plus environment (rotating wave approximation, see [3]). Under the assumption of such a perturbative interaction, energetics play a key role in determining the rate of decoherence processes. Such energy conserving decoherence has three possible forms: energy is supplied from the system to the environment (cooling), energy is supplied from the environment to the system (heating), or no energy is exchanged at all (nondissipative). Thus, even when the environment is a heat bath at zero temperature, cooling and, especially, nondissipative interactions can be a major source of decoherence.

The spirit of our approach to reducing decoherence is to force all *reasonable* decoherence mechanisms to be interactions which heat the system, such that at low bath temperatures decoherence is energetically suppressed. This is done by encoding into logical qubits which are the ground state of a particular engineered Hamiltonian. While all dissipative and dephasing processes act on the physical qubits, the only source of decoherence on the encoded qubits derives from nonenergy conserving decoherence pathways, which are by definition perturbatively weak. In particular, we will show the existence of a degenerate collective ground state of pairwise interacting two-level systems (qubits), which possesses the property that any local operation on an individual physical qubit must take the system out of this collective ground state. Quantum information can be encoded into the degeneracy of this ground state, to make an encoded qubit that is protected from any local decoherence which cannot overcome the established energy gap.

*Collective spin operations.*—Let $\mathcal{H}_n = (\mathbb{C}^2)^{\otimes n}$ be a Hilbert space of $n$ qubits, and let $\mathbf{s}_\alpha^{(i)}$ be the $\alpha$th Pauli spin operator acting on the $i$th qubit tensored with identity on all other qubits. We define the $k$th partial collective spin operators on the $n$ qubits, $\mathbf{S}_\alpha^{(k)} = \sum_{i=1}^k \mathbf{s}_\alpha^{(i)}$. The total collective spin operators acting on all $n$ qubits, $\mathbf{S}_\alpha^{(n)}$, form a Lie algebra $\mathcal{L}$ which provides a representation of the Lie algebra su(2): $[\mathbf{S}_\alpha^{(n)}, \mathbf{S}_\beta^{(n)}] = i\epsilon_{\alpha\beta\gamma}\mathbf{S}_\gamma^{(n)}$. Thus $\mathcal{L}$ can be decomposed in a direct product of irreducible representations (irreps) of su(2), $\mathcal{L} \simeq \bigoplus_{J=0,1/2}^{n/2}\bigoplus_{k=1}^{n_J}\mathcal{L}_{2J+1}$, where $\mathcal{L}_{2J+1}$ is the $(2J+1)$-dimensional irrep of su(2) which appears with a multiplicity $n_J$. If we let $(\mathbf{J}_d)_\alpha$ be the operators of the $d$-dimensional irrep of su(2), then there exists a basis for the total collective spin operators such that $\mathbf{S}_\alpha^{(n)} = \bigoplus_{J=0,1/2}^{n/2}\mathbf{I}_{n_J}\otimes(\mathbf{J}_{2J+1})_\alpha$. Corresponding to this decomposition of $\mathbf{S}_\alpha^{(n)}$, the Hilbert space $\mathcal{H}$ can be decomposed into states $|\lambda, J_n, m\rangle$ classified by quantum numbers labeling the irrep, $J_n$, the degeneracy index of the irrep, $\lambda$, and an additional internal degree of freedom, $m$. A complete set of commuting operators consistent with this decomposition and providing explicit values for these labels is given by $B_\alpha = \{(\vec{\mathbf{S}}^{(1)})^2, (\vec{\mathbf{S}}^{(2)})^2, \ldots, (\vec{\mathbf{S}}^{(n-1)})^2, (\vec{\mathbf{S}}^{(n)})^2, \mathbf{S}_\alpha^{(n)}\}$ [4]. Therefore a basis for the entire Hilbert space is given by $|J_1, J_2, \ldots, J_{n-1}, J_n, m_\alpha\rangle$, with $(\vec{\mathbf{S}}^{(k)})^2|J_1,\ldots,J_n,m_\alpha\rangle = J_k(J_k+1)|J_1,\ldots,J_n,m_\alpha\rangle$ and $\mathbf{S}_\alpha^{(n)}|J_1,\ldots,J_n,m_\alpha\rangle = m_\alpha|J_1,\ldots,J_n,m_\alpha\rangle$. The degeneracy index $\lambda$ of a particular irrep having total collective spin $J_n$ is completely specified by the set of partial collective spin eigenvalues $J_k$, $k<n$: $\lambda \equiv (J_1,\ldots,J_{n-1})$. This degeneracy is simply due to the $(n_J)$ different possible ways of constructing a spin $J_n$ out of $n$ qubits. In Fig. 1 we present a graphical method for understanding this degeneracy of the irreps. The internal quantum number $m_\alpha$ is the total spin projection along axis $\alpha$.

The $|\lambda, J_n, m\rangle$ states have a particular clean property for decoherence mechanisms which couple collectively to the system. Quantum information encoded into the degeneracy $|\lambda\rangle$ of these states is immune to collective decoherence. This information inhabits a decoherence-free (noiseless) subsystem [4–7]. Noncollective or local errors can still adversely affect decoherence-free subsystems [8]. In this paper we consider the action of independent errors acting on a code derived from decoherence-free states and we show that these errors can be suppressed by suitable construction of the energy spectrum. The decoherence-free property of the encoded states is retained in our approach. However, the method we present here deals with independent errors: as such, it can be used to reduce these errors irrespective of the existence of collective decoherence.

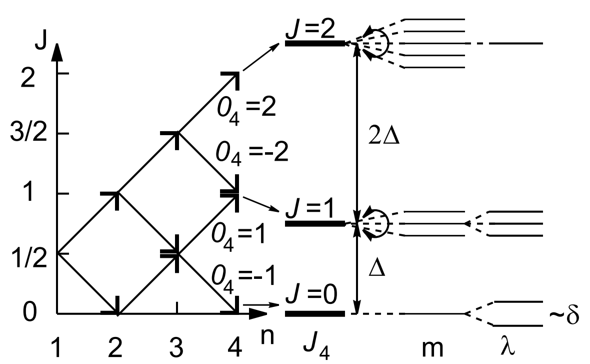


FIG. 1. Diagram showing formation of the $|\lambda, J_n, m\rangle$ states. The degeneracy index $\lambda$ of a given $J$ irreducible representation can be found by counting the number of paths which start with a spin-1/2 particle and which build up a total spin of $J$ using standard addition of angular momenta. Thus, each path in this figure, starting from $n=1$, $J_1 = 1/2$, is in one-to-one correspondence with a degeneracy index $\lambda$ of a given $J$ irrep. $O_n$ is the eigenvalue of $\mathbf{O}_n$ for the final step of this pathway. The allowed $\Delta J$ transitions are shown as double-ended arrows between the energy levels of $\mathbf{H}_0^{(4)}$ (see text for definition). Shown on the right are the $\lambda$ and $m$ degeneracies of the $J_4$ levels. The energy difference $\delta$ corresponding to a computation on the supercoherent qubit will split the $\lambda$ degeneracy.

*Collective Hamiltonian.*—The Hamiltonian $\mathbf{H}_0^{(n)} = \frac{\Delta}{2}(\vec{\mathbf{S}}^{(n)})^2$ has eigenvalues $\frac{\Delta}{2}J_n(J_n+1)$, with corresponding eigenstates $|\lambda, J_n, m_\alpha\rangle$. Thus the (possibly degenerate) ground state of such a Hamiltonian is given by the lowest $J_n$ states for a particular $n$. For $n$ even, these states have $J_n = 0$, and for $n$ odd they have $J_n = 1/2$. Furthermore, $\mathbf{H}_0^{(n)}$ can be constructed from two-qubit interactions alone: $\mathbf{H}_0^{(n)} = \frac{\Delta}{2}(\sum_{i\neq j=1}^n \vec{\mathbf{s}}^{(i)} \cdot \vec{\mathbf{s}}^{(\mathbf{j})} + \frac{3n}{4}\mathbf{I})$. Thus we see that $\mathbf{H}_0^{(n)}$ is nothing more than the Heisenberg coupling $\vec{\mathbf{s}}^{(i)} \cdot \vec{\mathbf{s}}^{(j)}$ acting with equal magnitude between every pair of qubits ($\mathbf{I}$ is an irrelevant energy shift).

*Effect of single-qubit operators.*—$\mathbf{H}_0^{(n)}$ has a highly degenerate spectrum, with energies determined by $J_n$. To determine the effect of single-qubit operations on these states, first consider the effect of a single-qubit operation on the $n$th qubit, $\mathbf{s}_\alpha^{(n)}$. Since $[\mathbf{s}_\alpha^{(n)}, (\vec{\mathbf{S}}^{(k)})^2] = 0$ for $k<n$, we see that $\mathbf{s}_\alpha^{(n)}$ cannot change the degeneracy index $\lambda$ of a state $|\lambda, J_n, m_\alpha\rangle$. Let $\mathbf{O}_n = -\frac{1}{4}\mathbf{I} + (\vec{\mathbf{S}}^{(n)})^2 - (\vec{\mathbf{S}}^{(n-1)})^2$ (defined for $n>1$). $\mathbf{O}_n$ determines which final step is taken in the addition from qubit $n-1$ to qubit $n$ (Fig. 1). If the final step from $J_{n-1}$ to $J_n$ was taken by adding 1/2, then the eigenvalue of $\mathbf{O}_n$ will be $O_n = J_{n-1} + \frac{1}{2}$, while, if it was taken by subtracting 1/2, then $O_n = -(J_{n-1} + \frac{1}{2})$. It is convenient to replace $(\vec{\mathbf{S}}^{(n)})^2$ by $\mathbf{O}_n$ in our set of commuting operators, which can clearly be done while still maintaining a complete set. We can then replace the quantum number $J_n$ by $O_n$, to obtain the basis $|\lambda, O_n, m_\alpha\rangle$. It is easy to verify that $\{\mathbf{O}_n, \mathbf{s}_\alpha^{(n)}\} = \mathbf{S}_\alpha^{(n)}$. If we examine the effect of $\mathbf{s}_\alpha^{(n)}$ on the basis $|\lambda, O_n, m_\alpha\rangle$ (where we have defined $m_\alpha$ in the orientation corresponding to $\mathbf{S}_\alpha^{(n)}$), we find that

$$(O_n' + O_n)\langle\lambda, O_n, m_\alpha|\mathbf{s}_\alpha^{(n)}|\lambda', O_n', m_\alpha'\rangle = m_\alpha \delta_{\lambda,\lambda'}\delta_{O_n,O_n'}\delta_{m_\alpha,m_\alpha'}. \quad (1)$$

Thus we see that the only nonzero matrix elements occur when $O_n' = O_n$ or $O_n' = -O_n$. From this it follows that the final step in the paths of Fig. 1 can either flip sign or must remain the same. Using the relation between the $O_n$ and $J_n$ bases, this results in the selection rules $\Delta J_n = \pm 1, 0$ for $\mathbf{s}_\alpha^{(n)}$ acting on states in the $|\lambda, J_n, m_\alpha\rangle$ basis. Note further that, if we had chosen a basis with $m_\beta$ instead of $m_\alpha$ in Eq. (1) ($\beta \neq \alpha$), the same selection rules would hold, but now the $m_\alpha$ components could be mixed by $\mathbf{s}_\beta^{(n)}$. In [4] it was shown that the exchange operation $\mathbf{E}_{ij} = \frac{1}{2}\mathbf{I} + 2\vec{\mathbf{s}}^{(i)}\vec{\mathbf{s}}^{(j)}$ which exchanges qubits $i$ and $j$ modifies only the degeneracy index $\lambda$ of the $|\lambda, J_n, m_\alpha\rangle$ basis. Because $\mathbf{s}_\alpha^{(j)} = \mathbf{E}_{jn}\mathbf{s}_\alpha^{(n)}\mathbf{E}_{jn}$, this implies that any single qubit operator $\mathbf{s}_\beta^{(i)}$ can therefore give rise to the mixing of both the spin projections $m_\alpha$ and the degeneracy indices $\lambda$.

These selection rules must be modified for the $J_n = 0$ states. $O_n = -1$ and $m_\alpha = 0$ for all $J_n = 0$ states, and any transitions between these states will therefore have a zero matrix element, i.e., $\langle\lambda, J_n = 0, m_\alpha|\mathbf{s}_\alpha^{(n)}|\lambda', J_n' = 0, m_\alpha'\rangle = 0$. Thus the transitions $\Delta J = 0$ are forbidden for $J_n = 0$, and $\mathbf{s}_\alpha^{(n)}$ *must* take $J_n = 0$ states to $J_n = 1$ states. Furthermore, since $\langle\lambda, J_n = 0, 0|\mathbf{s}_\alpha^{(n)}|\lambda', J_n' = 0, 0\rangle = 0$, the degeneracy index $\lambda$ for $J_n = 0$ states is not affected by any single-qubit operation.

To summarize, we have shown that any single-qubit operation $\mathbf{s}_\alpha^{(i)}$ enforces the selection rules $\Delta J_n = \pm 1, 0$ with *the important exception of* $J_n = 0$ *which must have* $\Delta J_n = +1$. The degenerate $J_n = 0$ states are therefore a quantum error detecting code for single-qubit errors [4,9], with the special property that they are also the ground state of a realistically implementable Hamiltonian [10]. The system Hamiltonian $\mathbf{H}_0^{(n)}$ has a ground state, for $n$ even, with the remarkable property that all single-qubit errors $\mathbf{s}_\alpha^{(i)}$ become dissipative heating errors.

*Coherence-preserving property.*—Figure 1 shows that for an even number of qubits the $J_n = 0$ ground state of $\mathbf{H}_0^{(n)}$ is degenerate. For $n = 4$ physical qubits, the ground state is 2-fold degenerate [6,9]. This degeneracy cannot be broken by any single-qubit operator, and single-qubit operations must take the $J_4 = 0$ states to $J_4 = 1$ states, as described above. The system Hamiltonian $\mathbf{H}_0^{(n)}$ has a ground state, for $n$ even, with the remarkable property that all single-qubit errors $\mathbf{s}_\alpha^{(i)}$ become dissipative heating errors. We will call this robust ground state a supercoherent qubit. If each qubit couples to its own individual environment, we expect that the major source of decoherence for these ground states will indeed be the processes which take the system from $J_4 = 0$ to $J_4 = 1$. What type of robustness should we expect for the supercoherent qubit? If the individual baths have a temperature $T$, then we expect the decoherence rate on the encoded qubit to scale at low temperatures as $\approx e^{-\beta\Delta}$, where $\beta = (kT)^{-1}$. At low temperatures there will thus be an exponential suppression of decoherence.

*Harmonic bath example.*—As an example of the expected coherence preservation, we consider a quite general model of four qubits coupling to four independent harmonic baths. The unperturbed Hamiltonian of the system and bath is $\mathbf{H}_0^{(4)} \otimes \mathbf{I} + \mathbf{I} \otimes \sum_{i=1}^4 \sum_{k_i} \hbar\omega_{k_i} \mathbf{a}_{k_i}^\dagger \mathbf{a}_{k_i}$, where $\mathbf{a}_{k_i}^\dagger$ is the creation operator for the $i$th bath mode with energy $\hbar\omega_{k_i}$. The most general linear coupling between each system qubit and its individual bath is $\sum_{i=1}^4 \sum_{k_i} \sum_\alpha \mathbf{s}_\alpha^{(i)} \otimes (g_{i,\alpha}\mathbf{a}_{k_i} + g_{i,\alpha}^* \mathbf{a}_{k_i}^\dagger)$. According to the selection rules described above we can write $\mathbf{s}_\alpha^{(i)} = \sum_{(p,q)\in S} \mathbf{A}_{i,\alpha}^{(p,q)} + \text{H.c.}$, where $\mathbf{A}_{i,\alpha}^{(p,q)\dagger}$ takes states $J_n = p$ to $J_n = q$ (and acts on $\lambda$ and $m_\alpha$ in some possibly nontrivial manner), and $S$ is the set of allowed transitions $S = \{(0,1), (1,2), (1,1), (2,2)\}$. In the interaction picture, after making the rotating-wave approximation [3], we find

$$\mathbf{V}(t) = \sum_{i,\alpha,k_i,(p,q)\in S} g_{i,\alpha}^* e^{-i[(\Delta/\hbar)f(p,q)-\omega_{k_i}]t} \mathbf{A}_{i,\alpha}^{(p,q)} \mathbf{a}_{k_i}^\dagger + g_{i,\alpha} e^{i[(\Delta/\hbar)f(p,q)-\omega_{k_i}]t} \mathbf{A}_{i,\alpha}^{(p,q)\dagger} \mathbf{a}_{k_i}, \quad (2)$$

where $f(p,q) = q(q+1) - p(p+1)$. Coupling to thermal environments of the same temperature, under quite general circumstances (Markovian dynamics, well-behaved spectral density of field modes), we are led to a master equation (see, for example, Ref. [3]),

$$\frac{\partial\rho}{\partial t} = \sum_{i,\alpha,(p,q)\in S} \gamma_{i,\alpha}^{(p,q)} \mathcal{L}_{i,\alpha}^{(p,q)}[\rho] + \gamma_{i,\alpha}^{(p,q)} \mathcal{L}_{i,\alpha}^{(p,q)}[\rho], \quad (3)$$

with $\mathcal{L}_{i,\alpha}^{(p,q)}[\rho] = ([\mathbf{A}_{i,\alpha}^{(p,q)}\rho, \mathbf{A}_{i,\alpha}^{(p,q)\dagger}] + [\mathbf{A}_{i,\alpha}^{(p,q)}, \rho\mathbf{A}_{i,\alpha}^{(p,q)\dagger}])$. The only operators which act on the supercoherent qubit are $\mathbf{A}_{i,\alpha}^{(0,1)}$. The relative decoherence rates satisfy $\gamma_{i,\alpha}^{(0,1)} \propto n(T)$, where $n(T) = [\exp(\beta\Delta) - 1]^{-1}$ is the thermal average occupation number. Thus we see that, as predicted, the supercoherent qubit decoheres at a rate which decreases exponentially as $kT$ decreases below $\Delta$.

Finally, we note that there are additional two-qubit errors on the system which can break the degeneracy of the supercoherent ground state. Such terms will arise in higher order perturbation theory and will result in a reduced energy gap of $\frac{g^2}{\Delta}$. These terms will produce decoherence rates $\mathcal{O}(g^2/\Delta^2)$ smaller than the $\mathcal{O}(g^2)$ single-qubit decoherence rates obtained without supercoherent encoding. In the perturbative regime, $g \ll \Delta$, this small factor therefore represents the limit to the protection offered by supercoherence.

*Universal quantum computation.*—In order to be useful for quantum computation, the supercoherent qubits should allow for universal quantum computation. Extensive discussion of universal quantum computation on qubits encoded in decoherence-free subsystems has been given in [4,9] (see also [11,12]), where it was shown that computation on these encoded states can be achieved by turning on Heisenberg couplings between neighboring physical qubits. This means that we need to add extra Heisenberg couplings to the supercoherent Hamiltonian $\mathbf{H}_0^{(4)}$. For a single supercoherent qubit these additional Heisenberg couplings can be used to perform any SU(2) rotation, i.e., an encoded one-qubit operation. In the present scheme one would like this additional coupling to avoid destroying the energy gap which suppresses decoherence. This can be achieved if the strength of the additional couplings, $\delta$, is much less than the energy gap, i.e., $\delta \ll \Delta$. The trade-off between the decoherence rate and the speed of the one-qubit operations can be quantified by calculating the gate fidelity $F \propto \delta e^{\beta(\Delta-\delta)}$. $F$ quantifies the number of operations which can be done within a typical decoherence time of the system. For small $\delta$ the gates are slower, while for larger $\delta$ the gap is smaller, resulting in a trade-off. $F$ is maximized for $\delta_0 = kT$. At this maximum, $F$ is still exponentially enhanced for lower temperatures. In particular, $F|_{\delta=\delta_0} \propto \beta^{-1}e^{\beta\Delta}$.

Of more concern for the present scheme is how to perform computation between two encoded supercoherent qubits. It can be shown that, by using only Heisenberg couplings, a nontrivial two-encoded qubit gate cannot be done without breaking the degeneracy of the $\mathbf{H}_0^{(4)}$ Hamiltonian on the two sets of four qubits. This can be circumvented by considering a joint Hamiltonian of the eight qubits, $\mathbf{H}_0^{(8)}$. This Hamiltonian has a ground state which is 14-fold degenerate, including the tensor product states of the degenerate ground state of the $\mathbf{H}_0^{(4)}$ Hamiltonian. The universality constructions previously presented in [4,9] can then easily be shown to never leave the ground state of this combined system.

Having shown how to perform quantum computation on the encoded qubits, it is also apparent that the supercoherent qubit will suffer decoherence when there is a lack of control of the Heisenberg interactions used either in constructing $\mathbf{H}_0^{(n)}$ or in performing a computation. Unless the magnitude of fluctuations in the Heisenberg interaction is large in comparison to the bath temperature, the resistance of the supercoherent qubit to local decoherence is, however, unaffected by these errors. Supercoherence, then, represents a method for eliminating the single-qubit decoherence process when superior two-qubit Hamiltonian control is possible [13].

*Separation of control and decoherence.*—A question which naturally arises is how the supercoherent qubits differ from encoding into a degenerate or nearly degenerate ground state of a single physical system. For example, one could encode information into the nearly degenerate hyperfine levels of an atomic ground state. There are essentially two differences between such a scheme and the supercoherent qubit. The first difference lies in the fact that the degenerate ground state of a single quantum system can interact with its environment in such a way that the coherence of the state is lost without the bath supplying energy to the system, i.e., nondissipatively. However, such a mechanism cannot affect a supercoherent qubit, because all $\mathbf{s}_\alpha^{(i)}$ interactions have been shown above to supply energy from the bath to the system. A second difference lies in the efficiency of manipulation of the system. If a nearly degenerate ground state is used for quantum computation, there is a trade-off between the speed of a single qubit gate and the decoherence rate. The limit on the speed of a supercoherent gate is, on the other hand, related to the temperature of the bath, with an error rate per quantum operation that scales in an exponentially favorable fashion. Supercoherent qubits therefore obtain a separation between *controlled* manipulation and *uncontrolled* decoherence, by making the control mechanisms two-body interactions which the single-qubit local decoherence cannot affect.

*Implementation in quantum dot grids.*—The technological difficulties in building a supercoherent qubit are daunting but we believe within the reach of present experiments. In particular, these coherence-preserving qubit states appear perfect for solid state implementations of a quantum computer using quantum dots [14]. Related decoherence-free encodings on three-qubit states were recently shown to permit universal computation with the Heisenberg interaction alone in [12]. The main new requirement for the supercoherent encoding, which allows the additional exponential suppression of decoherence not naturally achieved in decoherence-free states, is the construction of $\mathbf{H}_0^{(4)}$ and $\mathbf{H}_0^{(8)}$. $\mathbf{H}_0^{(4)}$ can be implemented by a two-dimensional array with Heisenberg couplings between all four qubits. $\mathbf{H}_0^{(8)}$ poses a more severe challenge, since the most natural geometry for implementing this Hamiltonian is eight qubits at the vertices of a cube with couplings between all qubits. Such structures should be possible in quantum dots by combining lateral and vertical coupling schemes. Finally, estimates of the strength of the Heisenberg coupling in the quantum dot implementations are expected to be on the order of 0.1 meV [14]. Thus we expect that at temperatures below 0.1 meV $\approx$ 1 K, decoherence should be suppressed for such coupled dots by encoding into the coherence-preserving states proposed here.

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