Symmetry-Breaking Induced Transport in the Vicinity of a Magnetic Island

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It is shown that in the vicinity of a magnetic island, the symmetry of the equilibrium magnetic-field strength B is broken due to the finite width of the islands. The magnitude of this broken symmetry is of the order of $(\delta B/B)^{1/2}$, where δB is the perturbed magnetic-field strength. This leads to enhanced plasma transport. The symmetry-breaking induced-transport flux in tokamaks with islands is calculated.

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Magnetic islands are ubiquitous in magnetically confined plasmas either produced in laboratories or exist in nature. They play an important role in fusion plasmas by their effect on plasma confinement. For example, a low m neoclassical magnetic island limits plasma beta, which is the ratio of plasma pressure to magnetic field pressure, in fusion grade plasmas [1–3]. Here m is the poloidal mode number. When a magnetic island is present, the equilibrium symmetry is broken. The broken symmetry in the magnetic field strength B is usually ignored, however. This is because the perturbed magnetic field strength δB of magnetic islands is small when compared with the equilibrium value of B. The symmetry breaking effect on B is thought to be of the order of $(\delta B/B)^2 \ll 1$. However, if B is not spatially uniform, e.g., $B = B(x)$, with x the radial variable, the symmetry breaking effect in B is of the order of $B'(x_0)(\Delta x)/B$. Here, x_0 is the position of the singular layer, prime denotes d/dx, and Δx is the width of the island. Because Δx is proportional to $(\delta B/B)^{1/2}$, the symmetry-breaking effect becomes much more important than has been perceived previously. This broken symmetry in B tends to enhance plasma transport. Here, we calculate symmetry-breaking induced-transport flux in tokamaks with an island to illustrate this effect.

The magnetic field $\mathbf{B}$ in a tokamak is modified by the presence of a magnetic island to become $\mathbf{B} = I\nabla\zeta + \nabla\zeta \times \nabla(\psi + \delta\psi)$, where ζ is the toroidal angle, $I = RB_t$, B_t is the toroidal magnetic strength, ψ is the unperturbed equilibrium poloidal flux function, and $\delta\psi = \tilde{\psi}\cos m\alpha$ is the perturbed poloidal flux function due to the presence of the island, with $\tilde{\psi}$ the amplitude of the perturbed poloidal flux, $\alpha = \theta - \zeta/q_s$ the helical angle, θ the poloidal angle, and q_s the safety factor at the radius of the O point of the island. The island winds around the torus helically. To see how a magnetic island modifies B on a perturbed magnetic surface, we first consider the variation of B on an equilibrium magnetic surface, which is, in a large aspect ratio tokamak,

$$B/B_0 = 1 - \varepsilon\cos\theta\,,$$

where B_0 is B on the magnetic axis, i.e., the origin of the open circles in Fig. 1, $\varepsilon = r/R \ll 1$ is the inverse aspect ratio, r is the local minor radius relative to the magnetic axis, and R is the major radius. Note that the reason that B has a $\cos\theta$ dependence on a magnetic surface, which is approximately a circle in this simple model, is because the plasma column has a finite radius, as can be seen from Fig. 1. When an island is present, the magnetic surface is distorted as shown in the lower shaded ellipse in Fig. 1. In the vicinity of an island, around the shaded region in Fig. 1, the magnetic surface is perturbed because $r = r_s + (r - r_s)$ with r_s the mode rational surface where the singular layer resides, or the center of the shaded region in Fig. 1, i.e., the O point of the island, and $(r - r_s)$ is deviation of the magnetic surface from the circle which is $r - r_s = \pm r_w(\bar{\Psi} + \cos m\alpha)^{1/2}$. Here the island width $r_w = \sqrt{2q_s^2\tilde{\psi}/q_s' B_0 r_s}$, $\bar{\Psi}$ is the normalized helical flux function $\bar{\Psi} = -\Psi/\tilde{\psi}$, $\Psi = \psi - \chi/q$ is the helical flux function, χ is the toroidal flux function, q is the safety factor, and $q_s' = dq/dr$ at $r = r_s$ [4]. Thus, in the vicinity of a magnetic island, the magnetic field

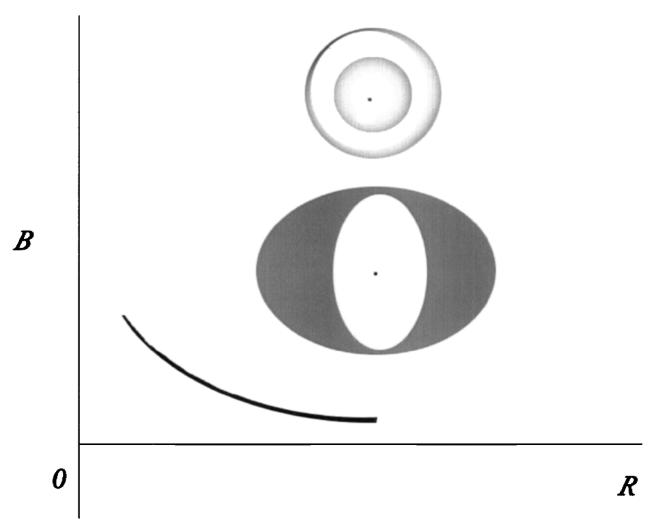

FIG. 1. The concentric circles denote equilibrium magnetic surfaces. The lower shaded ellipse represents the distorted magnetic surface due to an $m = 2$ island which is represented by the shaded region. The solid curve denotes the $1/R$ dependence of B.

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strength B has the following form:

$$B/B_0 = 1 - \left[\frac{r_s}{R} \pm \frac{r_w}{R}(\bar{\Psi} + \cos m\alpha)^{1/2}\right]\cos\theta\,. \quad (1)$$

The perturbed radial component of the magnetic field is of the order of $\delta B \approx m\tilde{\psi}/(Rr)$. This modification on B is of the order of $(\delta B)^2/B^2$, while the modification due to the distortion of the magnetic surface calculated in Eq. (1) is of the order of $(L_s q_s/mR)^{1/2}(\delta B/B)^{1/2} > (\delta B)^2/B^2$. Here, $L_s = q/(dq/dr)$. Thus, we neglect $(\delta B)^2/B^2$ modification in Eq. (1). From Eq. (1), it is clear that toroidal symmetry in B is broken in tokamaks by the perturbed magnetic surfaces due to the existence of the island. Even though we demonstrate the symmetry breaking in a tokamak with an island, it is obvious the principle is applicable to other plasma configurations with islands as well. Note that an island tends to increase the magnetic equilibrium dimension by 1. For example, tokamak equilibrium is two dimensional, i.e., B is a function of (r, θ). A magnetic island introduces one extra dimension in the perturbed equilibrium.

The symmetry breaking is especially significant for low m islands. For an $m = 2$ mode, r_w/r_s can be of the order of 10%. This is similar to a rippled tokamak with toroidal ripple strength of the order of a few percent in the core region caused by the finite numbers of toroidal field coils. With this magnitude of symmetry breaking perturbation in B, plasma transport can be enhanced over standard neoclassical values [5–7]. The fundamental mechanism for the enhanced transport in tokamaks with islands is that toroidally trapped particles, i.e., banana particles or particles trapped in the $\varepsilon\cos\theta$ variation of B on a magnetic surface, drift off the flux surface as a result of the nonvanishing bounce averaged radial drift velocity due to the broken symmetry in B.

To illustrate the enhanced transport, we calculate the transport fluxes in the presence of an island in the collisionless banana regime, i.e., the regime where the collision frequency is infrequent enough to allow the toroidally trapped particles to complete their collisionless trajectories. To this end, we solve drift kinetic equation [6]

$$v_\parallel \mathbf{n} \cdot \nabla f + \mathbf{v}_d \cdot \nabla f = C(f), \quad (2)$$

where $v_\parallel$ is the particle speed parallel to $\mathbf{B}$, $\mathbf{n} = \mathbf{B}/B$ is the magnetic field unit vector, $\mathbf{v}_d$ is the particle drift velocity, f is the particle distribution function, and $C(f)$ is the Coulomb collision operator. We adopt $(\Psi, \alpha, \theta, E, \mu)$ as independent variables with $E = Mv^2/2 + e\Phi$ the particle energy, M the mass, Φ the electrostatic potential, e the electric charge, $\mu = Mv_\perp^2/2B$ the magnetic moment, v the particle speed, and $v_\perp$ the perpendicular (to $\mathbf{B}$) speed. The helical flux function Ψ satisfies $\partial\Psi/\partial\psi = 1 - q/q_s$ and $\partial\Psi/\partial\alpha = \partial(\delta\psi)/\partial\alpha$, where q is the safety factor at r. We assume that the radial characteristic scale length of $\delta\psi$ is larger than the width of the island. In this coordinate system, Eq. (2) becomes

$$v_\parallel \mathbf{n}_0 \cdot \nabla\theta \frac{\partial f}{\partial\theta} + v_\parallel \mathbf{n}_0 \cdot \nabla\theta\left(1 - \frac{q}{q_s}\right)\frac{\partial f}{\partial\alpha} + \mathbf{v}_d \cdot \nabla f = C(f), \quad (3)$$

where $\mathbf{n}_0$ is the unit vector of the unperturbed equilibrium magnetic field $\mathbf{B}_0$. The second term on the left of Eq. (3) describes the slow drift due to the variation of q in the vicinity of the mode rational surface. We assume this slow drift speed is of the same order as $\mathbf{v}_d \cdot \nabla\alpha$.

To solve Eq. (4), we expand it in terms of small parameter $|\mathbf{v}_d|/v_t$, where v_t is the particle thermal speed, and assume the collision frequency is of the same order as the drift frequency. The lowest order equation is then

$$v_\parallel \mathbf{n}_0 \cdot \nabla\theta \frac{\partial f_0}{\partial\theta} = 0\,. \quad (4)$$

The solution to Eq. (4) is $f_0 = f_0(\Psi, \alpha)$. The distribution function f_0 is determined by the next order equation

$$v_\parallel \mathbf{n}_0 \cdot \nabla\theta \frac{\partial f_1}{\partial\theta} + v_\parallel \mathbf{n}_0 \cdot \nabla\theta\left(1 - \frac{q}{q_s}\right)\frac{\partial f_0}{\partial\alpha} + \mathbf{v}_d \cdot \nabla f_0 = C(f_0), \quad (5)$$

where f_1 is the first order distribution function. Note that parallel particle speed is $v_\parallel = \sqrt{2(E - \mu B - e\Phi)/M}$. Because B is a function of (α, θ), particles can be trapped in magnetic wells.

We are interested in particles that are trapped in the toroidal magnetic well. Because toroidal symmetry is broken, trapped particles will drift off the flux surface. We bounce average Eq. (5), i.e., operate with $\oint d\theta/v_\parallel \mathbf{n}_0 \cdot \nabla\theta$ to annihilate the $\partial f_1/\partial\theta$ term for both circulating and trapped particles, to obtain

$$\left(1 - \frac{q}{q_s}\right)\frac{\partial f_0}{\partial\alpha} H(\mu - \mu_c) + \langle \mathbf{v}_d \cdot \nabla\alpha\rangle \frac{\partial f_0}{\partial\alpha} + \langle \mathbf{v}_d \cdot \nabla\Psi\rangle \frac{\partial f_0}{\partial\Psi} = \langle C(f_0)\rangle, \quad (6)$$

where the angular brackets denote bounce averaging *without* dividing by $\oint d\theta/v_\parallel \mathbf{n}_0 \cdot \nabla\theta$, μ_c denotes the trapped and circulating boundary in the phase space, $H(\mu - \mu_c) = 1$ for circulating particles with $\mu < \mu_c$, and $H(\mu - \mu_c) = 0$ for trapped particles with $\mu > \mu_c$. Note the slow drift in the α direction associated with $(1 - q/q_s)$ vanishes for trapped particles.

The distribution function $f_0(\Psi, \alpha)$ is determined by a subsidiary expansion of Eq. (6). We adopt an optimal ordering by assuming the drift frequency in the α direction is of the same order of the collision frequency. The leading order equation is then

$$\left(1 - \frac{q}{q_s}\right)\frac{\partial f_{00}}{\partial \alpha} H(\mu - \mu_c) + \langle \mathbf{v}_d \cdot \nabla\alpha\rangle \frac{\partial f_{00}}{\partial \alpha} = \langle C(f_{00})\rangle, \quad (7)$$

where f_{00} is the leading order distribution function of the subsidiary expansion. A solution to Eq. (7) is the Maxwellian distribution $f_{00} = f_M(\Psi)$. Because f_{00} is a function of Ψ only, the equilibrium electrostatic potential Φ with islands is also a function of Ψ only. The next order equation is

$$\left(1 - \frac{q}{q_s}\right)\frac{\partial f_{01}}{\partial \alpha} H(\mu - \mu_c) + \langle \mathbf{v}_d \cdot \nabla\alpha\rangle \frac{\partial f_{01}}{\partial \alpha} + \langle \mathbf{v}_d \cdot \nabla\Psi\rangle \frac{\partial f_M}{\partial \Psi} = \langle C(f_{01})\rangle, \quad (8)$$

where f_{01} is the first order distribution of the subsidiary expansion.

Equation (8) is the basic equation to be solved to calculate the enhanced transport in the presence of an island in the low collision frequency regime. For simplicity, we will only display the solution in the regime where the effective collision frequency is larger than the drift frequency in the α direction. In that limit, f_{01} is determined by the equation

$$\langle \mathbf{v}_d \cdot \nabla\Psi\rangle \frac{\partial f_M}{\partial \Psi} = \langle C(f_{01})\rangle. \quad (9)$$

Before we present the solution of Eq. (9), we discuss the bounced averaged radial drift velocity

$$\langle \mathbf{v}_d \cdot \nabla\Psi\rangle = 8\frac{I}{\Omega}\frac{\partial \Psi}{\partial \psi}\sqrt{\frac{\mu B_0}{M\Delta}}\left(\mathbf{E} - \frac{\mathbf{K}}{2}\right)\frac{\partial \Delta}{\partial \alpha}, \quad (10)$$

where Ω is the gyrofrequency, $\mathbf{K}$ and $\mathbf{E}$ are complete elliptic integrals of the first and second kind, respectively, and $\Delta = \varepsilon_s \pm \delta_w(\bar{\Psi} + \cos m\alpha)^{1/2}$ with $\varepsilon_s = r_s/R$ and $\delta_w = r_w/R$. The argument of the complete elliptic integrals is $\kappa^2 = (E - \mu B_0 - e\Phi + \mu B_0\Delta)/2\mu B_0\Delta$. For particles trapped in the toroidal magnetic well $\kappa^2 < 1$, and for circulating particles $\kappa^2 > 1$.

It is obvious that the bounce averaged radial drift velocity which results from the symmetry breaking in B is reflected in $\partial\Delta/\partial\alpha \neq 0$. The magnitude is proportional to the width of the island or $(\delta B/B)^{1/2}$. The angle dependence is roughly $\sin m\alpha$. Note that the factor $\partial\Psi/\partial\psi$ is purely geometrical and is due to the choice of Ψ as the radial variable. The radial scale of Ψ in terms of the equilibrium poloidal flux function ψ is of the order of $(q_s'/q_s)r_w$. With a pitch angle scattering Coulomb collision operator, Eq. (9) can be easily integrated once to obtain [8–10]

$$\frac{\partial f_{01}}{\partial \kappa^2} = \frac{\mu B_0}{\nu}\Delta\frac{\partial f_M}{\partial \Psi}\frac{I\mathbf{B}_0 \cdot \nabla\theta}{M\Omega B_0}\frac{\partial \Psi}{\partial \psi}\frac{\partial \Delta}{\partial \alpha} \times \frac{\int_0^{\kappa^2} d\kappa^2\,(2\mathbf{E} - \mathbf{K})}{[\mathbf{E} - (1 - \kappa^2)\mathbf{K}]}, \quad (11)$$

where ν is the collision frequency. We only need to know $\partial f_{01}/\partial\kappa^2$ to calculate flux surfaced averaged transport fluxes, e.g., the particle flux $\Gamma = \langle N\mathbf{V} \cdot \nabla\Psi\rangle_f$, to obtain

$$\Gamma = -\frac{C_1}{2}\frac{(I\mathbf{n}_0 \cdot \nabla\theta)^2}{M^{7/2}\Omega^2}\left(\frac{q_s'}{q_s}r_w\right)^2 m^2\delta_w^2\varepsilon_s^{3/2} \times \frac{F(\bar{\Psi})\sqrt{1 + \bar{\Psi}}}{\mathbf{K}(\kappa_f)}\int dW\, W^{5/2}\frac{1}{\nu}\frac{\partial f_M}{\partial \Psi}, \quad (12)$$

where N is plasma density, $\mathbf{V}$ is the flow velocity, $\langle\rangle_f$ denotes flux surface average, $C_1 = \int_0^1 d\kappa^2\{(\int_0^{\kappa^2} d\kappa^2\,(2\mathbf{E} - \mathbf{K}))^2/[\mathbf{E} - (1 - \kappa^2)\mathbf{K}]\} = 0.684$, $W = Mv^2/2$, $F(\bar{\Psi}) = \oint d\alpha\,(\sin m\alpha)^2(\Delta/\varepsilon_s)^{3/2}/\sqrt{\bar{\Psi} + \cos m\alpha}$, and $\kappa_f^2 = 2/(1 + \bar{\Psi})$, which is a parameter that separates inside and outside of the island separatrix. Note again the extra $[(q_s'/q_s)r_w]^2$ in the flux expression is due to the choice of the coordinate system. The flux calculated here is applicable only in the region outside the island separatrix, i.e., $\kappa_f^2 < 1$. The region inside the island separatrix is denoted by $\kappa_f^2 > 1$. The equilibrium gradient vanishes there ($\partial f_M/\partial\Psi = 0$). The function $F(\bar{\Psi}) = -(8\sqrt{2}/3) \times (\bar{\Psi}^2 - 1)Q_{1/2}^1 > 0$, when $\Delta/\varepsilon_s \approx 1$ where $Q_{1/2}^1$ is the associated Legendre function of the second kind. Because the flux is inversely proportional to the collision frequency, it can be larger than the standard neoclassical flux in high temperature fusion plasmas with a low m island. Of course, the $1/\nu$ scaling will not persist indefinitely as ν decreases. Eventually the drift motion in the α direction will limit the step size to the width of the drift orbits and the transport flux will scale linearly with the collision frequency. The expression for the heat flux is similar to the particle flux given in Eq. (12) except there is an extra factor $(W - 5T/2)$ in the $\int dW$ integral. Here, T is the plasma temperature. Comparing Eq. (12) with the toroidal ripple induced flux [9], we see qualitative differences. In the toroidal ripple induced flux, it is the particles trapped in the toroidal ripple that contribute to the flux. The radial motion is caused by the gradient of the toroidal magnetic field. For the magnetic island induced flux, it is the toroidally trapped particles that contribute to the flux. The radial motion results from the gradient of the magnetic field variation on the perturbed magnetic surface in the presence of the islands.

The heat conductivity based on the calculation here in terms of the equilibrium poloidal flux scales like $(I\mathbf{n} \cdot \nabla\theta/\Omega)^2 m^2\delta_w^2\varepsilon_s^{3/2}(v_t)^4/\nu$, where v_t is the particle thermal speed. Thus, the ratio of the symmetric breaking induced heat flux to that of the banana regime heat flux [5–7] is of the order of $(m\delta_w/\varepsilon\nu_*)^2$, where ν_* is the ion collisionality parameter which characterizes the onset of the banana regime when $\nu_* < 1$. In the hot plasma core, typical value of ν_* is of the order of 10^{-2}. For a 10 cm wide island located at $r = 50$ cm, δ_w/ε is $1/10$. Thus, for an $m = 2$ island, the ratio of the symmetry breaking induced heat flux can be larger than that of the banana regime heat flux and may be comparable to the anomalous transport typically present in hot tokamaks.

In summary, we have shown that magnetic islands can break the equilibrium symmetry in the magnetic field strength B. This symmetry breaking leads to enhanced transport in the vicinity of the island. We have calculated the enhanced transport flux in a tokamak with an island and shown that it can be significant in hot plasma core.

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[1] Z. Chang, J.D. Callen, E.D. Fredrickson, R.V. Budny, K.M. McGuire, M.C. Zarnstorff, and TFTR Group, Phys. Rev. Lett. **74**, 4663 (1995).
[2] R. Carrera, R.D. Hazeltine, and M. Kotschenreuther, Phys. Fluids **29**, 899 (1986).
[3] J.D. Callen, W.X. Qu, K.D. Siebert, B.A. Carreras, K.C. Shaing, and D.A. Spong, in *Plasma Physics and Controlled Nuclear Fusion Research, Kyoto, 1986* (IAEA, Vienna, 1987), Vol. 2, p. 157.
[4] R.D. Hazeltine and J.D. Meiss, *Plasma Confinement* (Addison-Wesley, Redwood City, California, 1992).
[5] A.A. Galeev and R.Z. Sagdeev, Sov. Phys. JETP **26**, 233 (1968).
[6] F.L. Hinton and R.D. Hazeltine, Rev. Mod. Phys. **48**, 239 (1976).
[7] S.P. Hirshman and D.J. Sigmar, Nucl. Fusion **21**, 1079 (1981).
[8] E.A. Freeman, Phys. Fluids **13**, 490 (1970).
[9] T.E. Stringer, Nucl. Fusion **12**, 689 (1972).
[10] K.C. Shaing and S.A. Hokin, Phys. Fluids **26**, 2136 (1983).