$1/f$ Noise and Extreme Value Statistics

T. Antal,[1] M. Droz,[1] G. Györgyi,[2] and Z. Rácz[2,3]

[1]*Département de Physique Théorique, Université de Genève, CH 1211 Genève 4, Switzerland*
[2]*Institute for Theoretical Physics, Eötvös University, 1117 Budapest, Pázmány sétány 1/a, Hungary*
[3]*Laboratoire de Physique Théorique, Bâtiment 210, Université de Paris-Sud, 91405 Orsay Cedex, France*

We study finite-size scaling of the roughness of signals in systems displaying Gaussian $1/f$ power spectra. It is found that one of the extreme value distributions, the Fisher-Tippett-Gumbel (FTG) distribution, emerges as the scaling function when boundary conditions are periodic. We provide a realistic example of periodic $1/f$ noise, and demonstrate by simulations that the FTG distribution is a good approximation for the case of nonperiodic boundary conditions as well. Experiments on voltage fluctuations in GaAs films are analyzed and excellent agreement is found with the theory.

It was about 70 years ago that a current-carrying resistor was first observed to exhibit voltage fluctuations with a power spectra nearly proportional to the inverse of the frequency [1]. Since then it has been shown that such a $1/f$ noise is present in an extraordinary variety of phenomena [2], examples being the light emission of white dwarfs [3], the flow of sand through an hourglass [4], the ionic current fluctuations in membranes [5], and the number of stocks traded daily [6]. Most of the $1/f$ fluctuations are Gaussian, although non-Gaussian cases are also known [2,5]. The universality of this scale invariant phenomenon led to suggestions that part of the explanations should come from a generic underlying mechanism. Despite a large body of works, however, such a mechanism has not been discovered yet.

In this Letter, we establish a connection between $1/f$ noise and extreme statistics that may provide a new angle at the generic aspect of the phenomena. Namely, we shall show that Gaussian $1/f$ power spectra in periodic systems imply that the distribution of fluctuations in the finite-size "width" of the signal is one of the extreme value distributions, the FTG distribution [7].

As we shall see, the FTG distribution emerges as a finite-size scaling function in the above connection. Thus the result can also be viewed as an interesting contribution to the gallery of nonequilibrium scaling functions that can be effectively used in investigating far from equilibrium processes. Indeed, imagine that a distribution function is measured in experiments (or simulations). Comparing this function with those in the theoretically built gallery (note the absence of fitting parameters), one can identify the relevant features of the underlying dynamics in the experimental (or model) system. Such an approach was initiated in connection with surface growth problems [8–11], and the results have been used to establish universality classes in rather diverse processes such as massively parallel algorithms [12] and the interface dynamics in the $d = 2$ Fisher equation [13]. This line of reasoning [14] has also led to a connection between the dissipation fluctuations in a turbulence experiment [15] and the interface fluctuations in the

$d = 2$ Edwards-Wilkinson model (XY model) [16,17,19] and, furthermore, it helped in a parameter-free analysis of the upper critical dimension of the Kardar-Parisi-Zhang equation [20].

Derivation of the FTG distribution follows the steps of a similar calculation for the width distribution of random-walk interfaces [8]. Let the time evolution of the physical quantity of interest in the interval $0 \leq t \leq T$ be given by $h(t)$. This time series is equivalent to a surface configuration with $h(t)$ being the height of the surface over a $d = 1$ dimensional substrate of length T with t being the coordinate along the substrate. The quantity of interest is the mean-square fluctuations of the surface (also called roughness or width-square of the surface) given by

$$w_2(h) = \overline{[h(t) - \bar{h}]^2}, \qquad (1)$$

where the overbar denotes the average of a function over t,

$$\bar{F} = \frac{1}{T} \int_0^T F(t)\, dt. \qquad (2)$$

Let us assume now that the path probability of a given time series $h(t)$ is known $\mathcal{P}[h(t)] \sim \exp[-S(h)]$. Then the probability distribution of the surface fluctuations, $P(w_2)$, can be expressed as a path integral [21]

$$P(w_2) = \int \mathcal{D}h(t)\delta(w_2 - [\overline{h^2} - \bar{h}^2])\mathcal{P}[h(t)]. \qquad (3)$$

We shall restrict the above functional integral to periodic paths $h(t) = h(t + T)$ and, in order to keep $\mathcal{P}$ normalizable, the integration is carried out with $\bar{h}$ kept fixed.

The next step is to introduce the generating function for the moments of $P(w_2)$:

$$G(s) = \int_0^\infty dy\, P(y)e^{-sy}. \qquad (4)$$

Substituting $\mathcal{P}[h(t)] \sim \exp[-S(h)]$ into (3) and evaluating the integral (4), we find the following functional integral

$$G(s) = \mathcal{N} \int \mathcal{D}h(t) \exp[-S(h) - s(\overline{h^2} - \bar{h}^2)], \qquad (5)$$

where $\mathcal{N}$ is a normalization constant to ensure $G(0) = 1$.

The key question now is how to choose S in the probability density functional. The mathematical representation of $1/f$ noise has been pioneered by Mandelbrot and Van Ness [22], and since then a quite intricate theory (involving, e.g., fractional derivatives) has emerged [23]. Here we propose that the path probability of a Gaussian, periodic, random phase, and perfectly $1/f$ noise with the dispersion being linear for all frequencies, can be described by the following simple action:

$$S = \sigma \sum_{n=-L}^{L} |n| |c_n|^2 = 2\sigma \sum_{n=1}^{L} n|c_n|^2, \quad (6)$$

where σ is a parameter setting the effective surface tension and the c_n-s are the Fourier coefficients of the signal

$$h(t) = \sum_{n=-(N-1)/2}^{(N-1)/2} c_n e^{2\pi i n t/T}, \qquad c_{-n} = c_n^{\star}. \quad (7)$$

Here $h(t)$ is given on N equidistant points ($t = k\Delta t$, $T = N\Delta t$), and we introduced the notation $L = (N - 1)/2$ with N assumed to be odd.

Using (6) the functional integral (5) can be written as

$$G(s) = \bar{\mathcal{N}} \int \mathcal{D}[c] \exp\left[- \sum_{n=1}^{L} 2(\sigma n + s) |c_n|^2 \right]. \quad (8)$$

The integrals over the real and imaginary parts of $c_n (n = 1, 2, \ldots, L)$ yield a simple form for the generating function once we have used the condition $G(0) = 1$ to determine the normalization constant $\tilde{\mathcal{N}}$ in (8):

$$G(s) = \prod_{n=1}^{L} \left(1 + \frac{s}{\sigma n} \right)^{-1}. \quad (9)$$

The moments of $P(w_2)$ can now be calculated and, in particular, one finds that the average of w_2 diverges for large L as

$$\langle w_2 \rangle = -\frac{dG}{ds}\bigg|_{s=0} = \frac{1}{\sigma} \sum_{n=1}^{L} n^{-1} \approx \frac{1}{\sigma} [\ln L + \gamma], \quad (10)$$

with $\gamma = 0.577\ldots$ being the Euler constant. On the other hand, the fluctuations of w_2 are finite

$$\langle w_2^2 \rangle - \langle w_2 \rangle^2 = \frac{a^2}{\sigma^2}; \qquad a = \frac{\pi}{\sqrt{6}}. \quad (11)$$

This means that the scaling variable $w_2/\langle w_2 \rangle$ usually considered [8] in cases when $\sqrt{\langle w_2^2 \rangle - \langle w_2 \rangle^2} \sim \langle w_2 \rangle$ is not the best choice since the distribution function reduces to a delta function. Just as in case of the fluctuations of the $d = 2$ EW interface [16], the nontrivial shape underlying the delta function can be made visible by introducing

$$x = \frac{w_2 - \langle w_2 \rangle}{\sqrt{\langle w_2^2 \rangle - \langle w_2 \rangle^2}}. \quad (12)$$

Then the $L \to \infty$ limit of the inverse Laplace transform of $G(s)$ yields $P(w_2)$ in the following scaling form:

$$\Phi(x) \equiv \sqrt{\langle w_2^2 \rangle - \langle w_2 \rangle^2}\, P(w_2)$$

$$= \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} e^{xs} \prod_{n=1}^{\infty} \frac{e^{s/an}}{1 + \frac{s}{an}}. \quad (13)$$

The infinite product in (13) is equal to $e^{(\gamma s/a)}\Gamma(1 + s/a)$, so the inverse Laplace transform can be evaluated using Euler's integral formula for the Γ function, and one obtains

$$\Phi(x) = a e^{-(ax+\gamma) - e^{-(ax+\gamma)}}. \quad (14)$$

This scaling function, shown in Fig. 1, is one of the central results of our paper. In $\Phi(x)$ one recognizes the FTG distribution [7] which is one of the three limiting forms of extreme value statistics.

Extreme statistics has been studied in many contexts and the FTG distribution emerges frequently [24]. A recent example in connection with surfaces is the study of the scaling behavior of the growth of the maximal relative height of a surface [25]. Since in most of these studies an extreme property is investigated, it is not entirely surprising to see the FTG distribution appearing. For $1/f$ noise, however, this is not the case. A simple quantity such as the mean-square fluctuations (roughness of the interface) is distributed according to extreme (FTG) statistics. Although we do not see a simple physical reason that necessitates this mathematical result, we speculate that it may be a key feature that underlies a unified treatment of systems displaying $1/f$ noise.

When trying to compare the scaling function $\Phi(x)$ with experimental results the following problem arises. An experimental signal is analized by moving a window of length L and building a histogram from the values of w_2 computed for the windows. The problem now is that the boundary conditions for the windows are not periodic and it is known that the boundary conditions affect the scaling
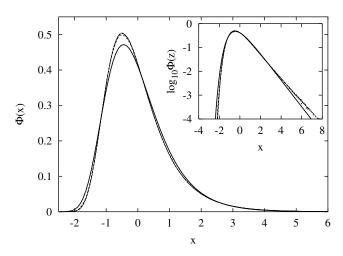

FIG. 1.   The analytical form (14) of the roughness distributions for a $1/f$ signal in the periodic case ($N = \infty$, solid line) as compared to the numerical result for the nonperiodic case for several values of N varying from 64 to 32 768 (dashed lines). The inset shows the same curves on semilog scale.

functions [26]. Thus two questions should be answered. First, is there a physical system with an effective action (6) of the $1/f$ noise where periodic boundary conditions are realized? Second, how sensitive is $\Phi(x)$ to the boundary conditions.

Let us begin with the first question by showing an example of such a system. We consider the steady-state fluctuations of a $d = 2$ Edwards-Wilkinson (EW) surface with the substrate taken to be an infinite plane. We draw a circle of radius R on the substrate and compute the probability density functional of a height configuration $h_B(\varphi)$ over this circle (parametrized by $0 \leq \varphi < 2\pi$). We shall find that the action in this functional is equal to that of the Gaussian $1/f$ noise given by (6).

In the $d = 2$ EW model, the steady-state fluctuations of the height of a surface $h(\vec{x})$ are characterized by the following free-field action:

$$S_2[h(\vec{x})] = \frac{\sigma}{2} \int |\nabla h(\vec{x})|^2 \, d^2\vec{x}, \qquad (15)$$

where the integral extends over a plane and $h(\vec{x})$ is bounded at infinity. Working with polar coordinates (r, φ) and specifying the height on a circle of radius R by $h(r = R, \varphi) = h_B(\varphi)$, the probability distribution functional of $h_B(\varphi)$ becomes

$$\hat{P}[h_B(\varphi)] = Z^{-1} \int_{h(\vec{x})|_R = h_B(\varphi)} \mathcal{D} h(\vec{x}) e^{-S_2[h(\vec{x})]}, \qquad (16)$$

where the functional integration is carried out with the boundary condition fixed at $r = R$, while Z is the same integral without restriction on the circle.

Since the action is quadratic in the field, the $h_B(\varphi)$ dependence of the above functional integral is contained in $\exp\{-S_c[h_B(\varphi)]\}$, where $S_c[h_B(\varphi)]$ is the extremal action for the given boundary condition. The field $h_c(r, \varphi)$ that extremizes $S_2[h]$ can be obtained by finding the solution of the Laplace equation that is finite both inside and outside the circle, and satisfies the boundary condition on it. Substitution of the resulting $h_c(r, \varphi)$ into (15) yields the classical action for the height fluctuations on the circle

$$S_c[h_B(\varphi)] = 2\sigma \sum_{n=1}^{\infty} n|h_n|^2, \qquad (17)$$

where the h_n-s are the Fourier amplitudes in $h_B(\varphi) = \frac{1}{\sqrt{2\pi}} \sum_n h_n \exp(in\varphi)$. Comparing (6) and (17), one finds that $S_c[h_B(\varphi)]$ is the action of a perfect Gaussian $1/f$ noise. Since $h_B(\varphi)$ is a periodic function, we have thus indeed obtained a physical realization of a periodic signal with $1/f$ noise.

Let us now turn to the question of how sensitive $\Phi(x)$ is to changes in the boundary conditions. Analytically this turns out to be a hard problem and only numerical calculations were performed. First, a periodic series of length $\tilde{N}$ having $1/f$ power spectrum was produced using an appropriately filtered Gaussian white noise. Next, the signal was divided into nonoverlapping segments of length

N and having determined the w_2-s, the histogram of w_2 and then the scaling function was built (using the same normalization as for the periodic boundary condition case). In order to obtain satisfactory precision we used $\tilde{N} = 2^{24}$ and averaged over 200 realizations of the periodic signals.

For $\tilde{N} = N$ the result for the periodic case is recovered while, in the $N \ll \tilde{N}$ limit, one finds that $\Phi(x)$ is independent of the size of the segments in a wide range of N values. The $\Phi(x)$ obtained in this $N \ll \tilde{N}$ limit will be considered as the scaling function for nonperiodic (or "experimental") boundary conditions. As one can see from Fig. 1, the distribution of nonperiodic signals deviates from that of the periodic case (FTG). The deviations, however, are small and mainly concentrated around the maximum of the function.

We now consider the case of voltage fluctuations in semiconductor films. The experiment was made by A. V. Yakimov and F. N. Hooge [27]. They considered n-type epitaxial GaAs films grown by molecular beam epitaxy. A noise-free current passed longitudinally through the film and different contacts were used as voltage probes both for longitudinal and transverse directions. Several time series with typically 163 840 points were obtained and the power spectrum was found to exhibit $1/f$ behavior roughly over two decades [27] of frequencies. We have reanalyzed these data by dividing the signal into segments of length $N = 32, 64, 128, 256$, and computing the distributions of the roughness for different N. The results are displayed on Fig. 2. One can see that the experimental data fit well with the theoretical curves (note that no fits are used in collapsing these functions) and that their precision is not good enough to distinguish between the periodic and nonperiodic cases.

In summary, our results suggests that extreme events and their statistics may underlie the physical and mathematical
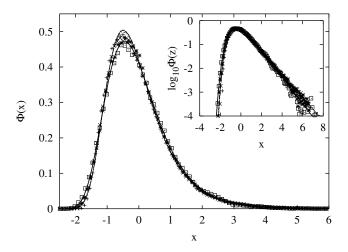

FIG. 2. Roughness distribution in the experiments on voltage fluctuations in resistors calculated for $N = 32$ (+), 64 ($\times$), 128 ($\ast$), and 256 ($\square$), compared to the analytical and numerical results shown in Fig. 1. The inset shows the same curves on a semilog scale.

description of Gaussian $1/f$ fluctuations. The next (rather difficult) task is clearly the identification of the relevant extreme events in a concrete physical system.

We should note that the possible importance of extreme statistics in scale invariant systems has also been discussed in recent works by Bramwell *et al.* [14–17]. They observe that the magnetization distribution of the $d = 2XY$ model at low temperatures is fit well by a generalized FTG distribution. They argue that this distribution function may also have relevance to other critical systems [17]. This suggestion is debated [18] and, indeed, a distribution function obtained by Gaussian approach may not be directly applicable to generic (inherently non-Gaussian) critical systems and, furthermore, the universality of the generalized FTG distribution should be limited by the dependence on boundary conditions (as shown in Fig. 1 for the $1/f$ noise). Nevertheless, these works may be important in pointing towards the possibility of nonconventional analysis of strongly correlated systems.

As a final remark, let us note that the apparent ubiquity of the $1/f$ noise is partly the result of loose terminology. Systems with $1/f^{\alpha}$ power spectrum are also said to exhibit $1/f$ noise if α is close to 1. The approach we used for the treatment of the pure $1/f$ noise can be extended to these systems and the dependence of the scaling function $\Phi(x)$ on α can be determined. A detailed discussion will be given in a forthcoming paper [28].

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