Mean-Field Approximation and Extended Self-Similarity in Turbulence

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Recent experimental discovery of extended self-similarity (ESS) was one of the most interesting developments, enabling precise determination of the scaling exponents of fully developed turbulence. A sufficient condition for extended self-similarity in a general dynamical system is derived in this paper. It is also shown that if the pressure-gradient contributions are expressed in terms of velocity differences in the mean-field approximation [V. Yakhot, Phys. Rev. E **63**, 026307 (2001)], then the ESS is a consequence of the Navier-Stokes equations.

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Scaling relations for velocity structure functions in isotropic and homogeneous turbulence are defined as

$$S_{n,m} = \langle [u(\mathbf{x}+\mathbf{r},t) - u(\mathbf{x},t)]^n [v(\mathbf{x}+\mathbf{r},t) - v(\mathbf{x},t)]^m \rangle = c_{n,m}(\mathcal{E}r)^{(n+m)/3}\left(\frac{r}{L_f}\right)^{\xi nm-(n+m)/3} \phi_{n,m}\left(\frac{r}{L_f},\frac{r}{\eta}\right), \tag{1}$$

where u and v are components of velocity field parallel and perpendicular to the displacement vector $\mathbf{r}$, respectively. The universality assumption implies the coefficients $c_{n,m} = O(1)$, independent of the Reynolds number (dissipation scale $\eta \approx L_f \mathrm{Re}^{-3/4}$). The dissipation rate $\mathcal{E} = \overline{(\partial_i u_j)^2} = \mathrm{const} = O(1)$ is equal to the power of external kinetic energy pumping. The shape of the structure functions (1) is an assumption, not following any rigorous theory. In the inertial range ($\frac{r}{L_f} \to 0, \frac{r}{\eta} \to \infty$) the scaling functions $\phi_{nm}(r) \to a_{n,m} = \mathrm{const}$, independent of the displacement r.

Both physical and numerical experiments show that the functions $\phi_{n,m}$ start deviating from the constant inertial range values at $r/\eta \approx 10$. Since one does not have theoretical expressions for $\phi_{n,m}$, accurate measurements of exponents $\xi_{n,m}$ in a fully developed turbulent flow requires an extremely wide range of variation of the displacement r which is possible only if the Reynolds number of a flow is huge. This problem is even more severe for numerical simulations of turbulence, where usually the wide inertial range is difficult to generate. It has been shown in a remarkable paper by Benzi *et al.* [1] that even in the medium (quite low, actually) Reynolds number flows, where (1) is hard to observe, the following relation [extended self-similarity (ESS)] holds:

$$S_{n,0}(r) = C_{n,m} S_{m,0}(r)^{\beta(n,m)}, \tag{2}$$

where $\beta(n,m) = \frac{\xi_n}{\xi_m}$. It is clear from (1) that if $c_{n,m}$ are Reynolds number independent, then the coefficients $C_{n,m}$ in (2) do not depend on the dissipation scale η (Reynolds number). Since the range of validity of expression (2) is much wider than that of (1), accurate determination of exponents $\beta(n,m)$ enables one to evaluate the exponents ξ_{nm} even in the not-too-high Reynolds number flows. Comparison of the exponents calculated this way with those measured in extremely high Re flows $[\beta(n,m) \approx \xi_{n,0}/\xi_{m,0}]$ was usually extremely good [2]. Since its discovery the relation (2) evolved into a major tool for experimental and numerical determination of the exponents ξ_{nm} [1–5]. The definition (2) was introduced and tested in Ref. [3]. Since $S_{3,0} \propto r$ in the inertial range, it is typically used in application of the ESS (2) for analysis of experimental data. It is shown below that extended self-similarity (2) can be derived self-consistently from the Navier-Stokes equations.

The following well-known relations will be useful below [6–7]:

$$\frac{dS_{2,0}(r)}{dr} = (d-1)[S_{0,2}(r) - S_{2,0}(r)], \tag{3}$$

$$6S_{1,2}(r) = \frac{d(rS_{3,0}(r))}{dr}, \tag{4}$$

and

$$S_{3,0} = -0.8r + 6\nu\frac{dS_{2,0}}{dr}. \tag{5}$$

The relation (3) is purely kinematic, reflecting properties of the divergence-free, statistically homogeneous fields. The dynamic relations (4) and (5) are the consequences of the Navier-Stokes equations for an incompressible, statistically homogeneous flow. The relation (3) involves only even-order moments only. The importance of (3)–(5) for what follows is that they couple $S_{2,0}$ and $S_{0,2}$ with $S_{3,0}$ and $S_{3,0}$. It has been shown recently that in a statistically isotropic, homogeneous, and incompressible flow, governed by the Navier-Stokes equations, the following equation can be rigorously derived in the limit $r/L_f \to 0$ where the forcing function can be neglected [8]:

$$\frac{\partial S_{2n,0}}{\partial r} + \frac{d-1}{r}S_{2n,0} - \frac{(d-1)(2n-1)}{r}S_{2n-2,2} = -(2n-1)\mathcal{P}_{x,2n-2} + (2n-1)\nu D_{u,2n-2}, \tag{6}$$

where

$$\mathcal{P}_{x,2n-2} = \overline{[p_{x'}(x') - p_x(x)](\Delta u)^{2n-2}}, \tag{7}$$

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$p_x = \partial_x p(x, y, x)$, and

$$D_{u,n} = \overline{[\nabla^2 u(x') - \nabla^2 u(x)](\Delta u)^n}. \quad (8)$$

These relations are exact even in the low-Reynolds-number statistically isotropic and homogeneous flows in the range $r/L_f \to 0$. The relations (6)–(8) can be viewed as a dynamic generalization of (3). Indeed, the left side of (6) is similar to the kinematic relation (3): the dynamics enters exclusively through pressure and the dissipation contributions. The dissipation term $D_{u,2n}(r) = O(1)$ and, thus, $\nu D_{u,2n} \to 0$ as $\nu \to 0$ in the inertial range. To prove this statement, we consider

$$\begin{aligned}(2n - 1)\nu D_{u,2n-2} = -(2n - 1)(2n - 2) \\ \times \overline{[\mathcal{E}_u(2) + \mathcal{E}_u(1)](\Delta u)^{2n-3}} \\ + \nu \partial_r^2 S_{2n-1,0}(r),\end{aligned} \quad (9)$$

where $\mathcal{E}_u = \nu \overline{(\partial u)^2}$. The second term in (9) disappears in the inertial range in the limit $\nu \to 0$. To estimate the first contribution, we write neglecting the subscript u

$$\overline{[\mathcal{E}(2) + \mathcal{E}(1)](\Delta u)^{2n-3}} \leq \sqrt{\overline{[\mathcal{E}(1) + \mathcal{E}(2)]^2}} S_{(4n-6)/2}(r). \quad (10)$$

Since in the inertial range ($r \to 0$), $\overline{[\mathcal{E}(1) + \mathcal{E}(2)]^2} \propto r^{-\mu}$ with $\mu \approx 0.2$, this term is negligibly small compared to the $O(S_{2n,0}/r)$ contributions to (3) for not too small moment number n, provided $\xi_{2n,0}$ "bends" strongly enough with n.

The fact that in the inertial range the dissipation contributions to (6) can be neglected does not mean that the even-order structure functions are not affected by the dissipation processes. Equation (6) is not closed and, as a result, the even-order moments are coupled to the dissipation contributions appearing in the equations for the odd-order moments. This will be discussed below.

To close the relation (6) one needs expressions for the pressure-velocity correlation functions. The mean-field approximation, introduced in [8], is a statement that the pressure-gradient difference is expressible in term of a quadratic form of velocity differences. Since $\langle \Delta p_y (\Delta u)^2 \rangle = \langle \Delta p_y (\Delta v)^2 \rangle = 0$, we are left with

$$\Delta p_x \approx \frac{a(\Delta u)^2 + b(\Delta v)^2}{r} + c\frac{d}{dr}(\Delta u)^2 + \dots . \quad (11)$$

The coefficients a, b, c, etc., are chosen so that $\overline{\Delta p_x} = \overline{\Delta p_x \Delta u} = \overline{\Delta p_x \Delta v} = 0$. We also have (see [8]) $\Delta p_y \propto \Delta u \Delta v / r$.

The mean-field approximation (11) for the pressure operator is the only expression, not involving noninteger powers of the displacement r and noninteger-order derivatives d^γ/dr^γ, thus consistent with the Navier-Stokes equations. Realizing that this is a mere plausibility argument in favor of (11), it is gratifying to notice that this approximation leads to homogeneous equations for the structure functions, opening a possibility of anomalous scaling. Recent experimental data [4] gave a reasonably good support to the mean-field approximation.

Let us assume that $S_{2n} = S_{2n}(S_{2m})$ where m is an arbitrary number. This assumption is nontrivial since, in principle, the moment S_{2n} can also depend on the displacement r and dissipation scale η (Reynolds number). Substituting this into (6) gives

$$\frac{\partial S_{2n,0}}{\partial S_{2m,0}} = \frac{(d - 1)S_{2n,0} - (d - 1)(2n - 1)S_{2n-2,2} + (2n - 1)r\mathcal{P}_{x,2n-2} - (2n - 1)\nu r D_{u,2n-2}}{(d - 1)S_{2m,0} - (d - 1)(2m - 1)S_{2m-2,2} + (2m - 1)r\mathcal{P}_{x,2m-2} - (2m - 1)\nu r D_{u,2m-2}}. \quad (12)$$

The relation (2) holds if the right side of (12) is equal to $\frac{\xi_{2n,0} S_{2n,0}}{\xi_{2m,0} S_{2m,0}}$. Again, the relations (12) are exact everywhere as long as $r/L_f \to 0$.

Equations (6) and (12) are not closed since we do not have the relations coupling $S_{2n,0}$ with $S_{2n-2,2}$. We know that in the dissipation range, $r/\eta \to 0$, the functions $S_{2n,0} \propto S_{2n-2,2}$, and $\xi_{2n,0} = 2n$, while in the inertial range the correlation functions are characterized by the nontrivial exponents (1). In principle, based on [8], we can easily write equations for $S_{2n-2,2}$. However, they involve the correlation functions $S_{2n-4,4}$, etc.

Now we ask a central question: Consider a relatively low Reynolds number flow, so that the dissipation contributions to (6) and (12) cannot be neglected and the functions $\phi_{2n,0}(0, \frac{r}{\eta})$ vary with the displacement r. What is the structure of the theory preserving (2) but strongly violating the inertial range scaling $S_{2n,0} \propto r^{\xi_{2n,0}}$? At the top of the dissipation range $r/\eta \approx 1$–10 the scaling functions, violating the inertial range scaling are not small [see (1)]. For $2m = 2$ Eq. (12) simplifies to

$$\frac{\partial S_{2n,0}}{\partial S_{2,0}} = \frac{(d - 1)S_{2n,0} - (d - 1)(2n - 1)S_{2n-2,2} + (2n - 1)r\mathcal{P}_{x,2n-2} - (2n - 1)\nu r D_{u,2n-2}}{(d - 1)(S_{2,0} - S_{0,2})}, \quad (13)$$

where, by virtue of (3), in incompressible, isotropic, and homogeneous turbulence $(d - 1)(S_{0,2} - S_{2,0}) = -r\frac{dS_{2,0}}{dr}$. Both dissipation and pressure contributions do not appear in the denominator of (13). The relation (13) is possible only in an interval where the numerator in (13) is equal to $-r\frac{dS_{2n,0}}{dr}$.

Substituting this into (13) and using the scaling form (1) gives

$$\frac{\partial S_{2n,0}}{\partial S_{2,0}} = \frac{\frac{dS_{2n,0}}{dr}}{\frac{dS_{2,0}}{dr}} = \frac{\xi_{2n,0} S_{2n,0}}{\xi_{2,0} S_{2,0}} \frac{1 + \frac{x}{\xi_{2n,0}\phi_{2n,0}(x)} \frac{d\phi_{2n,0}(x)}{dx}}{1 + \frac{x}{\xi_{2n,0}\phi_{2,0}(x)} \frac{d\phi_{2,0}}{dx}}. \tag{14}$$

By assumption, $S_{2n} = S_{2n}(S_2)$, subject to "boundary condition" $S_{2n,0} = C_{2n,2} S_2^{\xi_{2n,0}/\xi_{2,0}}$ as $x \to \infty$. Solution to (14), satisfying these constraints, is $\phi_{2n,0} \propto \phi_{2,0}^{\xi_{2n,0}/\xi_{2,0}}$. Indeed, substituting this into the second Eq. (14), we are left with the differential equation, equivalent to the ESS (2) with the Reynolds-number-independent coefficient $C_{2n,2}$. One can see that the ESS is the only universal solution to Eq. (14), not involving any dependence on the Reynolds number (η). This result, which is a sufficient condition for the ESS, is independent of the model for the dissipation and pressure contributions.

In the inertial range, where the power laws for the structure functions are assumed to be valid, the dissipation contributions are negligibly small. In this range the numerator of Eq. (13) is equal to $-dS_{2n,0}/dr$, possible only if the mean-field approximations (11) were as accurate as the power laws themselves. It is interesting that, according to (6), the exponents $\xi_{2n,0}$ and $\xi_{0,2n}$ with $n > 1$ are not necessarily equal: the $O(S_{2n-2,2})$ contributions can be canceled by the corresponding terms in the pressure model (11).

The most-often-used expression for analysis of experimental data is $S_{2n,0} = S_{2n,0}(S_{3,0})$. To make a transition we have to express $\phi_{2,0}$ in terms of $\phi_{3,0}$. The function $\phi_{2,0}(x)$ can be readily self-consistently found from Eq. (5). The inertial range calculations [8] and both numerical and physical experiments [2] give $\xi_{2,0} \approx 0.7$ and $\phi_{2,0}(x) \approx a_{2,0} \approx 2.0$ (Kolmogorov constant $C_K \approx 1.6$). Substituting the ESS expression $S_{3,0}^{\xi_{2,0}} \propto S_{2,0}$ into (5) gives

$$\frac{6d\phi_{2,0}}{dx} = (0.8 - 0.3\phi_{2,0}^{1/\xi_{2,0}})x^{1-\xi_2} - 6\xi_2\phi_{2,0}/x, \tag{15}$$

where by definition of the dissipation scale $\nu\eta^{-2+\xi_2} = 1$ and $\mathcal{E} = 1$. The solution to this equation gives $\phi_{2,0}(x)$, gently approaching $a_{2,0} \approx 2$ as $x \to \infty$ very close to the experimental results by Dhruva *et al.* [9]. Noticeable deviations from this constant value start at $x \approx 30-50$ (at $x = 20$ the function $\phi_{2,0} \approx 1.65-1.7$). A similar equation was derived by Benzi *et al.* (Ref. [3]).

Now we can discuss the cases where the ESS is violated. In a strongly sheared wall flow one can introduce two Reynolds numbers. The first one is $\mathrm{Re} = \overline{U}L/\nu$ where L is the width of the channel (boundary layer, etc.) and $\overline{U}$ is a characteristic (mean) velocity. The second one ($\mathrm{Re} = u_* L/\nu$) is based on the friction velocity $u_*^2 = -\nu \frac{\partial U}{\partial y}|_{\mathrm{wall}}$. The dissipation rate $\mathcal{E} = O(\frac{U^3}{L}\mathrm{Re}_*^4/\mathrm{Re}^3)$ is a weak function of the Reynolds number (dissipation) scale. Thus, all structure functions, even if they can be written in a form (1), must involve the Re-dependent proportionality coefficients. This violates the assumptions leading to the ESS (2). Far enough from the wall, where $\mathcal{E} \approx \overline{U}^3/L$ with the Re-independent proportionality coefficient, one can expect the ESS to be valid.

In some sheared flows $1/L_s = \frac{\partial u}{\partial y}/u = O(1)$, the scaling functions also depend on $y/L_s \approx 1$. In these regions the y/L_s cannot be neglected and the simple derivation of the ESS (13) breaks down. The best example, illustrating this point, is the Kolmogorov flow driven by the forcing function $\mathbf{f} = (0, \cos L_s x)$. There we expect the ESS to hold in the vicinities of zeros of the local strain rate $\partial_x U_y \propto \sin(L_s x)$ and break down near local maxima (minima) of the strain rate. These conclusions agree with the experimental findings [10,11].

To conclude, a general statement, not related to a particular dynamical system, can be made (1) if the scaling relation (1) with the $O(1)$ coefficients $c_{n,m}$ is valid, and (2) if $S_{n,0} = S_{n,0}(S_{m,0})$ is independent on r and η, then $S_{n,0} = C_{n,m} S_{m,0}^{(\xi_{n,0}/\xi_{m,0})}$. This relation means that there exists only one dominating (dynamically relevant) scaling function $\phi_{i,j}$ and all others can be calculated in a simple way.

It can be shown that in the interval $x > 1$, the direct dissipation contribution to (13) is small. Then, the mean-field approximation justifies the assumption $S_{2m,0} = S_{2m,0}(S_{2,0})$. This leads to the ESS.

It follows directly from the Navier-Stokes equations that if the inertial range power laws exist, then

$$(d-1)S_{2n,0} - (d-1)(2n-1)S_{2n-2,2} + (2n-1)r\mathcal{P}_{x,2n-2} = -r\frac{dS_{2n,0}}{dr}. \tag{16}$$

This expression means that the inertial range pressure contribution to this equation must be $O(S_{2n,0})$ or $O(S_{2n-2,2})$. This justifies the mean-field approximation (11).

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[1] R. Benzi, S. Ciliberto, R. Tripiccione, C. Baudet, and S. Succi, Phys. Rev. E **48**, R29 (1993).
[2] K. R. Sreenivasan and B. Dhruva, Prog. Theor. Phys. **130**, 103 (1998).
[3] R. Benzi, S. Ciliberto, and G. Ruiz Chavarria, Physica (Amsterdam) **80D**, 385 (1995).
[4] S. Kurien and K. R. Sreenivasan, Phys. Rev. E (to be published).
[5] N. Cao, S. Chen, and Z. She, Phys. Rev. Lett. **86**, 3775 (2001).
[6] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **32**, 19 (1941).
[7] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, 1975), Vol. 2.
[8] V. Yakhot, Phys. Rev. E **63**, 026307 (2001).
[9] B. Dhruva and K. R. Sreenivasan (private communication).
[10] G. Stolovitzky and K. R. Sreenivasan, Phys. Rev. E **48**, 32 (1993).
[11] V. Borue (private communication).