One-Step Synthesis of Multiatom Greenberger-Horne-Zeilinger States

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(Received 6 July 2001; published 16 November 2001)

A scheme is proposed for generating maximally entangled Greenberger-Horne-Zeilinger (GHZ) atomic states for testing quantum nonlocality. In the scheme, three atoms are simultaneously sent through a nonresonant cavity in a vacuum state. They are initially in the same state, and thus there is no energy exchange in the process. The cavity-assisted collision results in a phase-shift which depends upon the collective atomic excitations. In principle, the scheme can be generalized to generate N-atom GHZ states. The scheme is insensitive to cavity decay and requires only one cavity, providing new prospects for testing fundamental aspects of quantum mechanics and for quantum information processing.

DOI: 10.1103/PhysRevLett.87.230404 PACS numbers: 03.65.Ta, 42.50.Dv

Entanglement of many particles is of fundamental interest to test quantum mechanics against local hidden theory [1,2]. Furthermore, it has practical applications in quantum information processing, such as quantum cryptography [3], computer [4], and teleportation [5]. In recent years, much attention has been directed to the generation of highly entangled states. Two-atom entangled states have been experimentally realized in microwave cavity QED [6] and ion traps [7,8]. Three-photon entanglement has also been observed and used to verify quantum nonlocality [9,10]. Four-particle entanglements have been demonstrated in ion traps [11] by using the technique proposed by Mølmer and Sørensen [12].

In microwave cavity QED, proposals have been presented for the generation of multiatom entangled states using atomic beams [13–16]. Recently, three-particle entanglement has been demonstrated within a cavity [17]. However, there have been no reports on the demonstration of quantum nonlocality using entanglement of three or more massive particles. We have proposed a scheme for the generation of two-atom entangled states within a nonresonant cavity [18] and three-atom Greenberger-Horne-Zeilinger (GHZ) states using two nonresonant cavities [19]. The advantage is that the cavity decay is suppressed during the procedure. Following the scheme of Ref. [18], an experiment has been reported, in which two Rydberg atoms crossing a nonresonant cavity are entangled by coherent energy exchange [20]. In this paper we propose an alternative scheme for the generation of three-atom GHZ states and test of quantum nonlocality without using Bell inequalities [2]. This time we require only a single cavity always in the vacuum state. The atoms are entangled by phase-shift depending on the number of collective atomic excitations, instead of energy exchange. Furthermore, in principle more atoms can be entangled by such a way.

We first consider the interaction of N two-level atoms with a single-mode cavity. The Hamiltonian is (assuming $\hbar = 1$)

$$H = H_0 + H_i, \tag{1}$$

where

$$H_0 = \omega a^+ a + \omega_0 \sum_{j=1}^{N} S_{z,j}, \tag{2}$$

$$H_i = g \sum_{j=1}^{N} (a^+ S_j^- + a S_j^+), \tag{3}$$

where $S_{z,j} = \frac{1}{2}(|e_j\rangle\langle e_j| - |g_j\rangle\langle g_j|)$, $S_j^+ = |e_j\rangle\langle g_j|$, and $S_j^- = |g_j\rangle\langle e_j|$, with $|e_j\rangle$ and $|g_j\rangle$ being the excited and ground states of the jth atom, a^+ and a are the creation and annihilation operators for the cavity mode, ω_0 is the atomic transition frequency, ω is the cavity frequency, and g is the atom-cavity coupling strength. In the case $\delta = \omega_0 - \omega \gg g\sqrt{\bar{n} + 1}$, with $\bar{n}$ being the mean photon number of the cavity field, there is no energy exchange between the atomic system and the cavity. The energy conversing transitions are between $|e_j g_k n\rangle$ and $|g_j e_k n\rangle$. The Rabi frequency λ for the transitions between these states, mediated by $|g_j g_k n + 1\rangle$ and $|e_j e_k n - 1\rangle$, is given by [19,21]

$$\lambda = \frac{\langle e_j g_k n|H_i|g_j g_k n + 1\rangle\langle g_j g_k n + 1|H_i|g_j e_k n\rangle}{\delta} + \frac{\langle e_j g_k n|H_i|e_j e_k n - 1\rangle\langle e_j e_k n - 1|H_i|g_j e_k n\rangle}{-\delta} = \frac{g^2}{\delta}. \tag{4}$$

Since the two transition paths interfere destructively, the Rabi frequency is independent of the photon number of the cavity mode. Then the effective Hamiltonian is

$$H_e = \lambda\left[\sum_{j=1}^{N} (|e_j\rangle\langle e_j| a a^+ - |g_j\rangle\langle g_j| a^+ a) + \sum_{j,k=1}^{N} S_j^+ S_k^-\right], \qquad j \neq k. \tag{5}$$

The first and second terms describe the photon-number dependent Stark shifts, and the third term describes the dipole coupling between the jth and kth atoms induced by the cavity mode. When the cavity mode is initially in the vacuum state $|0\rangle$, it will remain in the vacuum state throughout the procedure. Since $aa^+|0\rangle = |0\rangle$ and $a^+a|0\rangle = 0$, the effective

Hamiltonian reduces to

$$H_e = \lambda\left[\sum_{j=1}^{N} |e_j\rangle\langle e_j| + \sum_{j,k=1}^{N} S_j^+ S_k^-\right], \qquad j \neq k. \quad (6)$$

Agarwal *et al.* [22] have also derived the effective Hamiltonian for a collection of two-level atoms interacting dispersively with a single-mode electromagnetic field in a cavity suffering losses and shown that an atomic Schrödinger cat state can be generated from an atomic coherent state by the process. When the cavity mode is in the vacuum state the effective Hamiltonian of Ref. [22] is

$$H_e' = \eta S^+ S^- = \eta\left[\frac{N}{2}\left(\frac{N}{2} + 1\right) - S_z^2 + S_z\right], \quad (7)$$

where $S^+ = \sum_{j=1}^N S_j^+$, $S^- = \sum_{j=1}^N S_j^-$, $S_z = \sum_{j=1}^N S_{z,j}$, and $\eta = g^2\delta/(\kappa^2 + \delta^2)$, with κ being the cavity decay rate. It can be easily shown that $S^+S^- = \sum_{j=1}^N |e_j\rangle\langle e_j| + \sum_{j,k=1}^N S_j^+ S_k^-$, $j \neq k$. Under the assumption that the cavity decay rate κ is much smaller than the detuning δ so that κ can be neglected in the expression of η the effective Hamiltonian of Eq. (7) agrees with that of Eq. (6).

In order to generate a three-atom GHZ state we simultaneously send three two-level atoms across an initially vacuum cavity. Assume that each atom is initially in the state $(1/\sqrt{2})(|g_j\rangle + i|e_j\rangle)$. Then the state for the atomic system is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2^3}} \sum_{k=0}^{3} i^k |\phi_k\rangle, \quad (8)$$

where

$$|\phi_0\rangle = |g_1 g_2 g_3\rangle, \quad (9)$$

$$|\phi_1\rangle = (|e_1 g_2 g_3\rangle + |g_1 e_2 g_3\rangle + |g_1 g_2 e_3\rangle), \quad (10)$$

$$|\phi_2\rangle = (|e_1 e_2 g_3\rangle + |e_1 g_2 e_3\rangle + |g_1 e_2 e_3\rangle), \quad (11)$$

$$|\phi_3\rangle = |e_1 e_2 e_3\rangle. \quad (12)$$

$|\phi_k\rangle$ is the eigenstate of the effective Hamiltonian H_e,

$$H_e|\phi_k\rangle = k(4 - k)\lambda|\phi_k\rangle. \quad (13)$$

After an interaction time τ we have

$$|\psi(\tau)\rangle = \frac{1}{\sqrt{2^3}} \sum_{k=0}^{3} e^{-ik(4-k)\lambda\tau} i^k |\phi_k\rangle. \quad (14)$$

The cavity-assisted collision does not change the probabilities of each atom being in the ground and excited states. However, the phase of the atomic state is shifted by an amount depending upon the excitation k. With the choice $\lambda\tau = \pi/2$, we have

$$|\psi(\tau)\rangle = \frac{1}{\sqrt{2^3}} \sum_{k=0}^{3} i^{k^2} i^k |\phi_k\rangle = \frac{1}{4} \sum_{k=0}^{3} [e^{i\pi/4} + e^{-i\pi/4}(-1)^k] i^k |\phi_k\rangle$$
$$= \frac{1}{4} e^{i\pi/4}\left[\prod_{j=1}^{3}(|g_j\rangle + i|e_j\rangle) - i\prod_{j=1}^{3}(|g_j\rangle - i|e_j\rangle)\right]. \quad (15)$$

The time evolution has some similarity with that of a light field passing through an amplitude dispersive medium [23,24]. We can rewrite $|\psi(\tau)\rangle$ as

$$|\psi(\tau)\rangle = \frac{1}{\sqrt{2}} e^{i\pi/4}(|+_{1z}\, +_{2z}\, +_{3z}\rangle + |-_{1z}\, -_{2z}\, -_{3z}\rangle), \quad (16)$$

where

$$|+_{jz}\rangle = \frac{1}{\sqrt{2}}(|g_j\rangle + i|e_j\rangle), \quad (17)$$

$$|-_{jz}\rangle = \frac{1}{\sqrt{2}}(|e_j\rangle + i|g_j\rangle). \quad (18)$$

Since $|+_{jz}\rangle$ is orthogonal to $|-_{jz}\rangle$, $|\psi(\tau)\rangle$ is a GHZ state.

Assume that $|+_{jz}\rangle$ and $|-_{jz}\rangle$ are spin-up and spin-down states along the z axis. Then the spin-up and spin-down states along the x axis are

$$|+_{jx}\rangle = \frac{1}{\sqrt{2}}(|+_{jz}\rangle + |-_{jz}\rangle) = \frac{1+i}{2}(|g_j\rangle + |e_j\rangle), \quad (19)$$

$$|-_{jx}\rangle = \frac{1}{\sqrt{2}}(|+_{jz}\rangle - |-_{jz}\rangle) = \frac{1-i}{2}(|g_j\rangle - |e_j\rangle). \quad (20)$$

The spin-up and spin-down states along the y axis are

$$|+_{jy}\rangle = \frac{1}{\sqrt{2}}(|+_{jz}\rangle + i|-_{jz}\rangle) = i|e_j\rangle, \quad (21)$$

$$|-_{jy}\rangle = \frac{1}{\sqrt{2}}(|+_{jz}\rangle - i|-_{jz}\rangle) = |g_j\rangle. \quad (22)$$

We first analyze the predictions of local reality. We call elements of reality X_j, Y_j, Z_j with values 1 and -1 for spin-up and spin-down states along the x, y, and z axes, respectively. We then have the relation $Y_1Y_2X_3 = Y_1X_2Y_3 = X_1Y_2Y_3 = -1$ for the state $|\psi(\tau)\rangle$. We now consider the value of $X_1X_2X_3$. Local realism claims that any measurement on one particle should be independent of measurements on other particles. This leads to $X_1X_2X_3 = (Y_1Y_2X_3)(Y_1X_2Y_3)(X_1Y_2Y_3) = -1$. On the other hand, quantum mechanics predicts $X_1X_2X_3 = 1$. Thus, four experiments are sufficient for the demonstration of quantum nonlocality. If the results of $Y_1Y_2X_3$, $Y_1X_2Y_3$, and $X_1Y_2Y_3$ are all -1, the result of $X_1X_2X_3$ directly gives the test of quantum nonlocality.

We now turn to the problem of generating a four-atom GHZ state. Again assume that each atom is initially in the state $(1/\sqrt{2})(|g_j\rangle + i|e_j\rangle)$. Then the state for the atomic system is

$$|\psi(0)\rangle = \frac{1}{4}\sum_{k=0}^{4} i^k|\phi_k\rangle, \quad (23)$$

where

$$|\phi_0\rangle = |g_1g_2g_3g_4\rangle, \quad (24)$$

$$|\phi_1\rangle = (|e_1g_2g_3g_4\rangle + |g_1e_2g_3g_4\rangle + |g_1g_2e_3g_4\rangle + |g_1g_2g_3e_4\rangle), \quad (25)$$

$$|\phi_2\rangle = (|e_1e_2g_3g_4\rangle + |e_1g_2e_3g_4\rangle + |e_1g_2g_3e_4\rangle + |g_1e_2e_3g_4\rangle + |g_1e_2g_3e_4\rangle + |g_1g_2e_3e_4\rangle), \quad (26)$$

$$|\phi_3\rangle = (|e_1e_2e_3g_4\rangle + |e_1e_2g_3e_4\rangle + |g_1e_2e_3e_4\rangle + |e_1g_2e_3e_4\rangle), \quad (27)$$

$$|\phi_4\rangle = |e_1e_2e_3e_4\rangle. \quad (28)$$

After an interaction time τ of the four atoms with the cavity we have

$$|\psi(\tau)\rangle = \frac{1}{4}\sum_{k=0}^{4} e^{-ik(5-k)\lambda\tau} i^k|\phi_k\rangle. \quad (29)$$

With the choice $\lambda\tau = \pi/2$, we have

$$|\psi(\tau)\rangle = \frac{1}{4}\sum_{k=0}^{4} i^{k^2}|\phi_k\rangle = \frac{1}{4\sqrt{2}} e^{i\pi/4}\left[\prod_{j=1}^{4}(|g_j\rangle + |e_j\rangle) - i\prod_{j=1}^{4}(|g_j\rangle - |e_j\rangle)\right]. \quad (30)$$

We can rewrite $|\psi(\tau)\rangle$ as

$$|\psi(\tau)\rangle = \frac{1}{\sqrt{2}} e^{i\pi/4}[|+_{1z}\ +_{2z}\ -_{3z}\ -_{4z}\rangle - |-_{1z}\ -_{2z}\ +_{3z}\ +_{4z}\rangle], \quad (31)$$

where

$$|+_{jz}\rangle = \frac{1}{\sqrt{2}}(|g_j\rangle + |e_j\rangle), \qquad j = 1,2, \quad (32)$$

$$|+_{3z}\rangle = \frac{1}{\sqrt{2}}(|g_3\rangle - |e_3\rangle), \quad (33)$$

$$|+_{4z}\rangle = \frac{i}{\sqrt{2}}(|g_4\rangle - |e_4\rangle), \quad (34)$$

$$|-_{jz}\rangle = \frac{1}{\sqrt{2}}(|g_j\rangle - |e_j\rangle), \qquad j = 1,2, \quad (35)$$

$$|-_{jz}\rangle = \frac{1}{\sqrt{2}}(|g_j\rangle + |e_j\rangle), \qquad j = 3,4, \quad (36)$$

Since $|+_{jz}\rangle$ is orthogonal to $|-_{jz}\rangle$, $|\psi(\tau)\rangle$ is a four-atom GHZ state.

Assume that $|+_{jz}\rangle$ and $|-_{jz}\rangle$ are spin-up and spin-down states along the z axis. Then the spin-up and spin-down states along the x axis are

$$|+_{jx}\rangle = |g_j\rangle, \qquad j = 1,2,3, \quad (37)$$

$$|+_{4x}\rangle = \frac{1+i}{2}|g_4\rangle + \frac{1-i}{2}|e_4\rangle, \quad (38)$$

$$|-_{jx}\rangle = |e_j\rangle, \qquad j = 1,2,3, \quad (39)$$

$$|-_{4x}\rangle = \frac{i-1}{2}|g_4\rangle - \frac{1+i}{2}|e_4\rangle. \quad (40)$$

The spin-up and spin-down states along the y axis are

$$|+_{jy}\rangle = \frac{1+i}{2}|g_j\rangle + \frac{1-i}{2}|e_j\rangle, \qquad j = 1,2, \quad (41)$$

$$|-_{jy}\rangle = \frac{1-i}{2}|g_j\rangle + \frac{1+i}{2}|e_j\rangle, \qquad j = 1,2, \quad (42)$$

$$|+_{3y}\rangle = \frac{1+i}{2}|g_3\rangle + \frac{i-1}{2}|e_3\rangle, \quad (43)$$

$$|-_{3y}\rangle = \frac{1-i}{2}|g_3\rangle - \frac{1+i}{2}|e_3\rangle, \quad (44)$$

$$|+_{4y}\rangle = i|g_4\rangle, \quad (45)$$

$$|-_{4y}\rangle = -i|e_4\rangle. \quad (46)$$

For this four-atom GHZ state we have the relation $Y_1Y_2Y_3Y_4 = X_1Y_2X_3Y_4 = X_1Y_2Y_3X_4 = -1$. We now consider the value of $Y_1Y_2X_3X_4$. According to local realism $Y_1Y_2X_3X_4 = (Y_1Y_2Y_3Y_4)(X_1Y_2X_3Y_4)(X_1Y_2Y_3X_4) = -1$. On the other hand, quantum mechanics predicts $Y_1Y_2X_3X_4 = 1$. Thus, in this case four experiments are also sufficient for the demonstration of quantum nonlocality.

Assume that N atoms are sent through an initially vacuum cavity. Each atom is initially in the state $(1/\sqrt{2}) \times (|g_j\rangle + i|e_j\rangle)$. Then the state for the atomic system is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2^N}}\sum_{k=0}^{N} i^k|\phi_k\rangle, \quad (47)$$

where $|\phi_k\rangle$ is the sum of all $k(N+1-k)$ terms with k atoms in excited states. Applying the effective Hamiltonian H_e on each term we obtain $\lambda|\phi_k\rangle$ since H_e contains all possible exchange of excitation between two atoms. Thus we have

$$H_e|\phi_k\rangle = k(N+1-k)\lambda|\phi_k\rangle. \quad (48)$$

After an interaction time $\tau = \pi/(2\lambda)$ of the N atoms with the cavity we have

$$|\psi(\tau)\rangle = \frac{1}{\sqrt{2^N}} \sum_{k=0}^{N} e^{-ik(N+1-k)\pi/2} i^k |\phi_k\rangle = \frac{1}{\sqrt{2^{N+1}}} e^{i\pi/4} \left\{ \prod_{j=1}^{N} [|g_j\rangle + (-i)^N |e_j\rangle] - i \prod_{j=1}^{N} [|g_j\rangle - (-i)^N |e_j\rangle] \right\}. \quad (49)$$

A correlation inequality has been derived for such a state [25]. It has been shown that quantum mechanics violates this inequality by an amount growing exponentially with N.

We now give a brief discussion on the experimental matters. For the Rydberg atoms with principal quantum numbers 50 and 51, the radiative time is about $T_r \simeq 3 \times 10^{-2}$ s, and the coupling constant is $g = 2\pi \times 24$ kHz [26]. For a three-atom system, the damping time is $T_r' = T_r/3 \simeq 10^{-2}$ s. With the choice $\delta = 10g$, the required atom-cavity-field interaction time is on the order of $\pi\delta/(2g^2) \simeq 10^{-4}$ s. Then the time needed to complete the whole procedure is on the order of 10^{-3} s, much shorter than T_r'.

In conclusion, we have proposed a scheme for the generation of multiatom GHZ states and test of quantum nonlocality. The scheme has two distinct advantages. As in the scheme of our previous scheme [19], the cavity is always in the vacuum state and thus the cavity decay is suppressed. Another advantage is that a single atom-cavity interaction is sufficient to entangle many atoms and only one cavity is required. The process is essentially a cavity-assisted multiatom collision. In the previous case [18,20] the atoms are not initially in the same state and the entanglement arises from the coherence energy exchange. In the present scheme the atoms are initially in the same state, and thus there is no energy exchange in the process. The entanglement results from the phase-shift depending upon the number of collective atomic excitations. Our scheme opens a new way for engineering multiatom entanglement and testing quantum nonlocality. Our scheme might also be useful in quantum information processing, such as quantum cryptographic conference and quantum secret sharing [27]. Based on the techniques reported in Refs. [17] and [20] our scheme might be experimentally realizable.

This work was supported by the National Natural Science Foundation of China under Grant No. 60008003, Fujian Provincial Natural Science Foundation of China under Grants No. K20004 and No. F0110027, and Funds from Fuzhou University.

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