Multipartite Bound Entangled States that Violate Bell's Inequality

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We study the relation between distillability of multipartite states and violation of Bell's inequality. We prove that there exist multipartite bound entangled states (i.e., nonseparable, nondistillable states) that violate a multipartite Bell inequality. This implies that (i) violation of Bell's inequality is not a sufficient condition for distillability and (ii) some bound entangled states cannot be described by a local hidden variable model.

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Since the celebrated work by Bell [1], it is evident that quantum mechanics is not compatible with local realist theories. In fact, Gisin [2] and Gisin and Peres [3] showed that all bipartite entangled pure states violate the Bell-CHSH (Clauser-Horne-Shimony-Holt) inequality [4]. This rules out the existence of a local hidden variable (LHV) model which is capable of describing all statistical correlations predicted by quantum mechanics for such states. These results were readily generalized by Popescu and Rohrlich [5] to multipartite entangled pure states of arbitrary dimension.

However, for quantum systems in mixed states, the situation is much more complicated, and we still lack a complete classification of mixed states into "local" and "nonlocal" ones. Although structural knowledge about entanglement has increased rapidly in the last decade, many open questions remain to be answered, in particular concerning the relation of certain entanglement properties to the existence of LHV models.

While it is obvious that separable mixed states [6] can be described by a LHV model, Werner showed in his pioneering paper [7] that also a certain class of *entangled* mixed states — now called Werner states — do not violate any Bell-type inequality. This was done by explicitly constructing a LHV model which can simulate the results of any single (i.e., nonsequential) measurement performed at each side. However, some years later Popescu [8] realized that also sequential measurements can be considered, and showed that most of the Werner states exhibit violation of local realism if sequences of measurements are taken into account. This so-called "hidden nonlocality" is revealed by a sequence of two measurements, where the first measurement is used to select a certain subensemble of pairs — those pairs which produce a specific outcome — while the second measurement tests the Bell observable on the subensemble. If the subensemble does not satisfy Bell's inequality, then one concludes that the initial ensemble violates local realism. In fact, applying a similar reasoning to collective tests of particles [9], it was shown that all inseparable states of two qubits [10] violate local realism. More generally, all *distillable* [11,12] states violate local realism. It is an open problem whether violation of Bell's inequality, which seems to be a rather strong requirement, already implies distillability [13].

While in systems of two qubits all entangled states are distillable [10] (and thus violate local realism), this turned out to be false for higher dimensional systems. The Horodecki [16] discovered states which, although being nonseparable and thus entangled, are not distillable. Those states are called bound entangled. The role of bound entangled states is not entirely clear yet. Although they cannot be useful for any quantum information processing task directly (since they are nondistillable), it was nevertheless shown that they allow one to perform certain processes (e.g., quasidistillation) [17] which cannot be performed using local operations and classical communication alone. In particular, it is not known whether bound entangled states violate local realism or not. This paper is intended to shine some light on these questions.

We will, however, tackle this problem not in its bipartite setting but rather in the multipartite setting. To this aim, we consider N spatially separated parties. A generalization of the Bell-CHSH inequality to multipartite systems is due to Mermin [18], and was further developed, e.g., in Ref. [19] (see also Ref. [20] for a good overview on Bell inequalities and entanglement and Refs. [21–24] for some recent developments). Also the notion of separability and distillability can be readily generalized to multipartite systems (see, e.g., Refs. [25,26]). We have that an N-partite state ρ is called (fully) separable iff it can be written as a convex combination of (unnormalized) product states, i.e.,

$$\rho = \sum_i |a_i\rangle_{\text{party1}}\langle a_i| \otimes |b_i\rangle_{\text{party2}}\langle b_i| \otimes \ldots \otimes |n_i\rangle_{\text{party}N}\langle n_i| . \quad (1)$$

If ρ is not fully separable, it is entangled. Note that fully separable states are those which can be prepared locally. Because of the fact that many different kinds of multipartite pure state entanglement exist, there are various kinds of distillability [26]. We will, however, not distinguish between those possible kinds of entanglements but rather say that a state ρ is distillable, iff (by means of local operations and classical communication) out of an arbitrary number of identical copies of a state ρ *some* entangled pure state can be created. If no entangled pure state whatsoever can be created, ρ is nondistillable. As shown in Ref. [25] (see also [27]), there exist also multipartite bound entangled states, i.e., states which are not (fully) separable and hence entangled, but which are nondistillable. Here we investigate multipartite bound entangled states and consider the relation of their distillability to the existence of a LHV model. We show that there exist multipartite bound entangled states which violate Bell's inequality. This means that on the one hand violation of Bell's inequality does not imply distillability, and on the other hand some bound entangled states violate local realism.

In the remainder of this Letter, we consider specific N-qubit mixed states ρ_N acting on a Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$. We consider N parties, $A_1, \ldots, A_N$, at different locations, each of them possessing several qubits. The parties possess M identical copies of ρ_N, where M can be arbitrarily large. Thus the state of all qubits is described by the density operator $\rho_N^{\otimes M}$. This ensures that the parties can use distillation protocols [12] in order to obtain entangled pure states between some of them. The states ρ_N we consider are given by [28]

$$\rho_N = \frac{1}{N+1}\left(|\Psi\rangle\langle\Psi| + \frac{1}{2}\sum_{k=1}^{N}(P_k + \bar{P}_k)\right). \tag{2}$$

We have that $|\Psi\rangle$ is an N-party Greenberger-Horne-Zeilinger (GHZ) state [29],

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0^{\otimes N}\rangle + e^{i\alpha_N}|1^{\otimes N}\rangle), \tag{3}$$

with α_N being an arbitrary phase. We denoted by P_k a projector on the state $|\phi_k\rangle$, where $|\phi_k\rangle$ is a product state which is $|1\rangle$ for party A_k and $|0\rangle$ for all other parties, i.e., $|\phi_k\rangle = |0\rangle_{A_1}|0\rangle_{A_2}\ldots|1\rangle_{A_k}\ldots|0\rangle_{A_{N-1}}|0\rangle_{A_N}$. The projector $\bar{P}_k$ is obtained from P_k by replacing all zeros by ones and vice versa.

For the states ρ_N (2) we show that (i) the states ρ_N are bound entangled, i.e., nonseparable and nondistillable if the number of parties $N \geq 4$; (ii) the states ρ_N violate the Mermin-Klyshko inequality if the number of parties $N \geq 8$ and thus cannot be described by a LHV model.

We start out by showing (i), i.e., ρ_N is bound entangled. One readily verifies that $\rho_N^{T_{A_k}}$ is a positive operator $\forall\, k$, where T_{A_k} denotes partial transposition with respect to party A_k [30]. This already implies that ρ_N is nondistillable [26]. To see this, assume as the opposite that one can distill some bi- or multipartite entangled pure state. As shown below (Lemma 1), one can always create by means of local operations from any multipartite entangled pure state a maximally entangled *bipartite* pure state shared between two of the parties, say A_i and A_j. Thus the resulting state has nonpositive partial transposition with respect to parties A_i and A_j, while the initial state ρ_N has positive partial transposition. Because of the fact that by means of local operations and classical communication, one cannot change the positivity of the partial transposition [17], we have the desired contradiction; hence ρ_N is not distillable.

On the other hand, ρ_N can easily be seen to be entangled for $N \geq 4$, e.g., by observing that $\rho_N^{T_{A_k A_l}}$ is not a positive operator for $k \neq l$ and $N \geq 4$. This already implies that ρ_N is not fully separable, as positivity of all possible partial transpositions is a necessary condition for (full) separability in multipartite systems [25,31]. It remains to show the announced lemma.

Lemma 1: Given a single copy of an N-party entangled pure state $|\Phi\rangle$ of arbitrary dimension, one can always create with nonzero probability of success by means of local operations a maximally entangled bipartite pure state shared among some of the parties.

Proof: One may write $|\Phi\rangle$ in its Schmidt decomposition with respect to any party A_k. Since $|\Phi\rangle$ is entangled, there exists at least one party, say A_1, where one obtains a minimal number of two nonzero Schmidt coefficients (otherwise $|\Phi\rangle$ would be a N-party product state). That is,

$$|\Phi\rangle = \sum_{k=0}^{d}\lambda_k|k\rangle_{A_1}|\varphi_k\rangle_{A_2,\ldots,A_N}, \tag{4}$$

where $\langle k\,|\,k'\rangle = \langle\varphi_k\,|\,\varphi_{k'}\rangle = \delta_{kk'}$ and $d \geq 2$. For simplicity, let us assume that $d = 2$ and $\lambda_0 = \lambda_1 = 1/\sqrt{2}$ (this can always be accomplished by a filtering measurement in A_1, e.g., using $O_{A_1} = \lambda_1|0\rangle\langle 0| + \lambda_0|1\rangle\langle 1|$). Now, either (a) both $|\varphi_0\rangle$ and $|\varphi_1\rangle$ are product states or (b) at least one of them, say $|\varphi_0\rangle$, is entangled.

In case of (a), $|\varphi_0\rangle$ and $|\varphi_1\rangle$ have to be locally orthogonal in at least one location, say at A_2 (this is due to the fact that $\langle\varphi_0\,|\,\varphi_1\rangle = 0$), i.e., $|\varphi_0\rangle = |0\rangle_{A_2}|\chi_3\rangle_{A_3}\ldots|\chi_N\rangle_{A_N}$ and $|\varphi_1\rangle = |1\rangle_{A_2}|\tilde{\chi}_3\rangle_{A_3}\ldots|\tilde{\chi}_N\rangle_{A_N}$. There may exist l locations A_k, $l \leq N-2$ for which $|\chi_k\rangle = |\tilde{\chi}_k\rangle$. Each of the other parties A_k can apply a local filtering measurement of the form $O_{A_k} = |0\rangle\langle\chi_k'| + |1\rangle\langle\tilde{\chi}_k'|$, where $\{|\chi_k'\rangle, |\tilde{\chi}_k'\rangle\}$ is the biorthonormal basis to $\{|\chi_k\rangle, |\tilde{\chi}_k\rangle\}$, i.e., $\langle\tilde{\chi}_k\,|\,\chi_k'\rangle = \langle\chi_k\,|\,\tilde{\chi}_k'\rangle = 0$ and $\langle\tilde{\chi}_k\,|\,\tilde{\chi}_k'\rangle = \langle\chi_k\,|\,\chi_k'\rangle = 1$. One readily observes that this leads to the creation of a $N-l$ party GHZ state (3), from which, by means of local measurements in the basis $\{|+\rangle, |-\rangle\}$ where $|\pm\rangle = 1/\sqrt{2}(|0\rangle \pm |1\rangle)$ at the remaining locations, a maximally entangled bipartite pure state shared between any two out of the remaining $N-l$ parties can be created deterministically.

In case of (b), one measures in A_1 the projector $P_{A_1} = |0\rangle\langle 0|$ and is left with an entangled state of $N-1$ (or less) particles. This situation is similar to the one we started with; however, the number of entangled systems decreased. One proceeds in the same vein with the remaining systems until (a) applies, which happens in the worst case if only two entangled particles are left. Finally, one obtains at least a maximally entangled bipartite state shared among two of the parties which concludes the proof of the lemma.

Note that from Lemma 1 follows the nonexistence of a LHV model for *all* multipartite entangled pure states of arbitrary dimension which describes properly also sequences of measurements (see Ref. [5] for the stronger result of inconsistency with local realism even for single measurements per site). Following the reasoning of Popescu [8] (see also [32]) (adopted to the multipartite case), the violation of Bell's inequality of subensembles of states (obtained, e.g., by local filtering measurements) ensures that also the original ensemble violates local realism. Since, according to Lemma 1, from any multipartite entangled pure state a maximally entangled bipartite pure state can be created, which clearly maximally violates Bell's inequality, the claim follows.

We now turn to (ii) and show that ρ_N violates the Mermin-Klyshko inequality for $N \geq 8$. Let a_j, a_j' be two vectors on the unit sphere which indicate two possible measurement directions for party A_j. The corresponding observables are given by $O_j = \sigma_{a_j}, O_j' = \sigma_{a_j'}$. Up to a normalization factor, any k-qubit Bell inequality involving two observables per qubit can be written as

$$|\langle \mathcal{B}_k \rangle| \leq 1\,, \tag{5}$$

where $\mathcal{B}_k \equiv \mathcal{B}_k(a_1, a_2, \ldots, a_k, a_1', a_2', \ldots, a_k')$ is the corresponding Bell operator. We consider the Mermin-Kyshko inequalities [18,19] for N qubits, whose corresponding Bell operator is defined recursively as [24]

$$\mathcal{B}_k = \tfrac{1}{2}\mathcal{B}_{k-1} \otimes (\sigma_{a_k} + \sigma_{a_k'}) + \tfrac{1}{2}\mathcal{B}_{k-1}' \otimes (\sigma_{a_k} - \sigma_{a_k'})\,, \tag{6}$$

where $\mathcal{B}_k'$ is obtained from $\mathcal{B}_k$ by exchanging all the a_k and a_k'.

We choose the same measurement directions in all N locations, $\sigma_{a_j} = \sigma_x$ and $\sigma_{a_j'} = \sigma_y \;\forall\; j$, where σ_x, σ_y are Pauli matrices. One readily verifies that in this case $\mathcal{B}_N$ can be written as

$$\mathcal{B}_N = 2^{(N-1)/2}(e^{i\beta_N}|1^{\otimes N}\rangle\langle 0^{\otimes N}| + e^{-i\beta_N}|0^{\otimes N}\rangle\langle 1^{\otimes N}|)\,, \tag{7}$$

with $\beta_N \equiv \pi/4(N-1)$. Using that $\mathrm{tr}(\mathcal{B}_N P_k) = \mathrm{tr}(\mathcal{B}_N \bar{P}_k) = 0 \;\forall\; k$ and $\mathrm{tr}(\mathcal{B}_N |\Psi\rangle\langle\Psi|) = 2^{(N-1)/2}$ when fixing the phase $\alpha_N = \beta_N$ in $|\Psi\rangle$ (3), we have

$$\mathrm{tr}(\mathcal{B}_N \rho_N) = \frac{1}{N+1}\,2^{(N-1)/2}, \tag{8}$$

which fulfills $\mathrm{tr}(\mathcal{B}_N \rho_N) > 1$ iff $N \geq 8$. Thus the states ρ_N (2) with the choice $\alpha_N = \pi/4(N-1)$ violate the Mermin-Klyshko inequality for $N \geq 8$ as announced.

To conclude, we have shown that certain multipartite bound entangled states violate a multipartite Bell inequality. This implies that (i) violation of Bell's inequality is not a sufficient condition for distillability, and (ii) there does not exist a local hidden variable model for certain bound entangled states. Note that the states ρ_N for sufficiently large N violate the Mermin-Klyshko inequality directly, and no sequence of measurements, eventually performed on a tensor product of the states, is required to rule out the existence of a LHV model as in the case of hidden nonlocality. There remain a number of open problems concerning the relation of inseparability to the existence of a LHV model [33]. In particular, it is not known whether all bipartite bound entangled states, those with positive partial transposition as well as the conjectured ones with nonpositive partial transposition [34], can be described by a local hidden variable model or not.

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