Leverage Effect in Financial Markets: The Retarded Volatility Model

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(Received 9 January 2001; published 8 November 2001)

We investigate quantitatively the so-called "leverage effect," which corresponds to a negative correlation between past returns and future volatility. For individual stocks this correlation is moderate and decays over 50 days, while for stock indices it is much stronger but decays faster. For individual stocks the magnitude of this correlation has a universal value that can be rationalized in terms of a new "retarded" model which interpolates between a purely additive and a purely multiplicative stochastic process. For stock indices a specific amplification phenomenon seems to be necessary to account for the observed amplitude of the effect.

DOI: 10.1103/PhysRevLett.87.228701 PACS numbers: 89.65.Gh, 02.50.Le, 05.45.Tp, 87.23.Ge

Several "stylized facts" of financial markets, such as "fat tails" in the distribution of returns or long ranged volatility correlations, have recently become the focus of detailed empirical study [1–6]. Simple agent-based models have been proposed, with some degree of success, to explain these features [7]. Another well-known stylized fact is the so-called "leverage" effect, or volatility asymmetry, first discussed by Black [8,9], who observed that the amplitude of relative price fluctuations ("volatility") of a stock tends to increase when its price drops. This effect is particularly important for option markets [5,10,11]: option prices indeed reflect the fact that a negative volatility-return correlation induces a negative skew in the distribution of returns on longer time scales.

Although widely discussed in the economic and econometric literature [12–16], the volatility-return correlation has been less quantitatively investigated than the volatility clustering effect (volatility-volatility correlation). For example, one would like to know if the negative volatility-return correlation shows a long term dependence similar to that observed on the volatility-volatility correlation. Although various single correlation coefficients quantifying the leverage effect have been measured and discussed within GARCH-like models [13–15], the full temporal structure of this correlation has never been quantitatively investigated. The economic interpretation of this leverage effect is still controversial; a recent survey of the different models can be found in [14,15]. Even the causality of the effect is debated [12,14]: Is the volatility increase induced by the price drop or conversely do prices tend to drop after a volatility increase? According to Black, a price drop increases the risk of a company to go bankrupt, and its stock therefore becomes more volatile. On the contrary, one can argue [12,14] that an increase of volatility makes the stock less attractive, and therefore pushes its price down. At the end of this Letter, we discuss the "volatility feedback" mechanism [15–17].

In this Letter, we report an empirical study of this volatility-return correlation both for individual stocks and for stock indices. We find that correlations are between *future* volatilities and *past* price changes. For both stocks and stock indices, the volatility-return correlation is short ranged, with, however, a different decay time for stock indices (about 10 days) than for individual stocks (about 50 days). The amplitude of the correlation is also different, and is much stronger for indices than for individual stocks. We then argue that the leverage effect for stocks can be interpreted within a simple retarded model, where the absolute amplitude of price changes does not follow the price level instantaneously (as is assumed in most models of price changes, such as the geometric Brownian motion). Rather, absolute price changes are related to an average level of the past price. We then show that this model does not account properly for the data on stock indices, which seems to reflect a "paniclike" effect, whereby large price drops of the market, as a whole, trigger a significant increase of activity.

We will call $S_i(t)$ the price of stock i at time t, and $\delta S_i(t)$ the (absolute) daily price change: $\delta S_i(t) = S_i(t+1) - S_i(t)$. The relative (dimensionless) price change will be denoted as $\delta x_i(t) = \delta S_i(t)/S_i(t)$. The leverage correlation function which naturally appears in the calculation of the skewness of the distribution of price changes is

$$\mathcal{L}_i(\tau) = \frac{1}{Z}\langle[\delta x_i(t+\tau)]^2 \delta x_i(t)\rangle, \qquad (1)$$

which measures the correlation between price change at time t and a measure of the square volatility at time $t+\tau$. In the above formula, brackets refer to a time average and the coefficient Z is a normalization that we have chosen to be $Z = \langle \delta x_i(t)^2\rangle^2$ for reasons that will become clear below.

We analyzed a set of 437 U.S. stocks, constituent of the S&P 500 index and a set of 7 major international stock indices (S&P 500, NASDAQ, CAC 40, FTSE, DAX, Nikkei, and Hang Seng). Our dataset consisted of daily data ranging from January 1990 to May 2000 for stocks and from January 1990 to October 2000 for indices. We computed $\mathcal{L}_i$ both for individual stocks and for stock indices. The raw results were rather noisy. We therefore assumed that individual stocks behave similarly and averaged $\mathcal{L}_i$ over the 437 different stocks in our dataset to give $\mathcal{L}_S$, and over 7 different indices to give $\mathcal{L}_I$. The results are given in Figs. 1 and 2, respectively. The insets of these figures show $\mathcal{L}_S(\tau)$ and $\mathcal{L}_I(\tau)$ for negative values of τ. For stocks, the results are not significantly different from zero for $\tau < 0$. For indices, we find some significant *positive* correlations up to $|\tau| = 4$ days, which might reflect the fact that large daily drops [that contribute significantly to $\langle \delta x_i^2(t + \tau) \rangle$] are often followed by positive "rebound" days.

We now focus on positive values of τ. As can be seen from these figures, both $\mathcal{L}_S$ and $\mathcal{L}_I$ are significant and negative: price drops increase the subsequent volatility—this is the so-called leverage effect. These correlation functions can be fit rather well by single exponentials:

$$\mathcal{L}_{S,I}(\tau) = -A_{S,I} \exp\left(-\frac{\tau}{T_{S,I}}\right). \tag{2}$$

For U.S. stocks, we find $A_S = 1.9$ days and $T_S \simeq 69$ days, whereas for indices the amplitude A_I is significantly larger, $A_I = 18$ days, and the decay time shorter, $T_I \simeq 9$ days. This exponential decay should be contrasted with the very slow, power-law–like decay of the volatility correlation function, which cannot be characterized by a unique decay time [1–6].

Traditional models of asset fluctuations postulate that price changes are proportional to prices themselves. The price increment is therefore written as

$$\delta S_i(t) = S_i(t)\sigma_i(t)\epsilon_i(t), \tag{3}$$

where $\sigma_i(t)$ is the (time dependent) volatility and ϵ_i is a random variable with unit variance, independent of all past history. Equation (3) shows that price increments are at any time proportional to the current value of the price. Although it is true that, in the long run, price increments tend to be proportional to prices themselves, this is not reasonable on short time scales. Locally, prices evolve following buy or sell orders that are expressed as integer number of shares, and placed around the current price on a discrete grid of possible prices *expressed in dollars* (i.e., not in percent). The very mechanism leading to price changes is therefore not expected to vary continuously as prices evolve, but rather to adapt only progressively if prices are seen to rise (or decrease) significantly over a certain time window. The model we propose to describe this lagged response to price changes is to replace S_i in Eq. (3) by a moving average S_i^R over a certain time window [18]:

$$\delta S_i(t) = S_i^R(t)\sigma_i(t)\epsilon_i \qquad S_i^R(t) = \sum_{\tau=0}^{\infty} \mathcal{K}(\tau) S_i(t - \tau), \tag{4}$$

where $\mathcal{K}(\tau)$ is a certain averaging kernel, normalized to one, $\sum_{\tau=0}^{\infty} \mathcal{K}(\tau) \equiv 1$, and decaying over a typical time T. It will be more convenient to rewrite S_i^R as

$$\begin{aligned} S_i^R(t) &= \sum_{\tau=0}^{\infty} \mathcal{K}(\tau)\left[S_i(t) - \sum_{\tau'=1}^{\tau} \delta S_i(t - \tau') \right] \\ &= S_i(t) - \sum_{\tau'=1}^{\infty} \overline{\mathcal{K}}(\tau')\delta S_i(t - \tau'), \end{aligned} \tag{5}$$

where $\overline{\mathcal{K}}(\tau) = \sum_{\tau'=\tau}^{\infty} \mathcal{K}(\tau')$. Note that, from the normalization of $\mathcal{K}(\tau)$, one has $\overline{\mathcal{K}}(0) = 1$, independently of the specific shape of $\mathcal{K}(\tau)$. This will turn out to be crucial in the following discussion. For an exponential kernel, one finds $\overline{\mathcal{K}}(\tau) = \alpha^\tau$. From Eq. (5), one sees that the limit $\alpha \to 1$ corresponds to the case where $S_i^R(t)$ is a

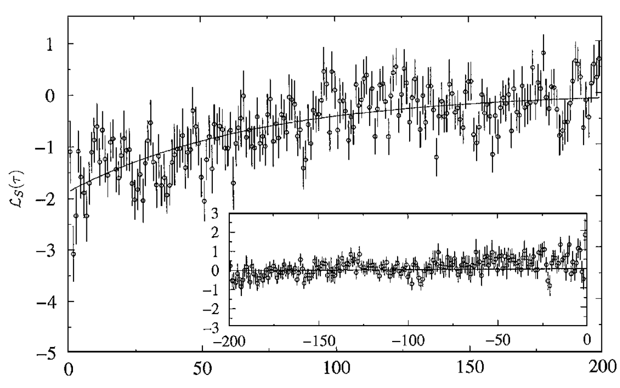

FIG. 1. Return-volatility correlation for individual stocks. Data points are the empirical correlation averaged over 437 U.S. stocks, and the error bars are two sigma error bars estimated from the interstock variability. The solid line shows an exponential fit [Eq. (2)] with $A_S = 1.9$ and $T_S = 69$ days. Note that $\mathcal{L}_S(0)$ is not far from the retarded model value -2. The inset shows the same function for negative times.

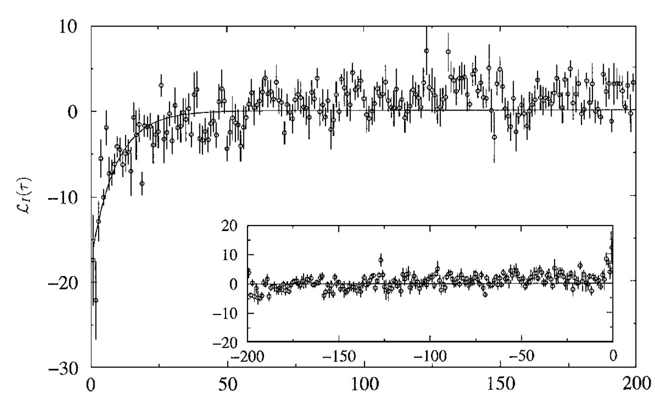

FIG. 2. Return-volatility correlation for stock indices. Data points are the empirical correlation averaged over 7 major stock indices, and the error bars are two sigma error bars estimated from the interindex variability. The solid line shows an exponential fit [Eq. (2)] with $A_I = 18$ and $T_I = 9.3$ days. The inset shows the same function for negative times.

constant, equal to the initial price $S_i(t = 0)$. Therefore, in this limit, our retarded model corresponds to an *additive* model, where the leverage effect would be trivial: a price drop obviously leads in that case to an increase in the *relative* fluctuations. The other limit $\alpha \to 0$ (infinitely small averaging time window) leads to $S_i^R(t) \equiv S_i(t)$ and corresponds to a purely *multiplicative* model. In the following, we will assume that the relative difference between S_i and S_i^R is small, which is the case when $\eta = \sigma\sqrt{T} \ll 1$ (i.e., small relative price changes over the horizon T). For constant volatility $[\sigma_i(t) = \sigma_i^0]$ and to first order in η, one finds [see Eq. (1)]

$$\mathcal{L}_i(\tau) = -2\overline{\mathcal{K}}(\tau). \tag{6}$$

An important prediction of this model is therefore that $\mathcal{L}_i(\tau \to 0) = -2$. Taking into account the volatility fluctuations would multiply the above result by a factor $\langle\sigma_i^2(t)\sigma_i^2(t + \tau)\rangle/\langle\sigma_i^2(t)\rangle^2 \geq 1$. As shown in Fig. 1, $\mathcal{L}_i(\tau \to 0)$ is close to (although indeed slightly below) the value -2 for individual stocks. We confirmed this finding by analyzing a set of 500 European stocks and 300 Japanese stocks, again in the period 1990–2000. We again found an exponential behavior with a time scale on the order of 40 days and, more importantly, initial values of $\mathcal{L}_i$ close to the retarded model value -2: $A_S = 1.96$ days and $T_S = 38$ days for European stocks and $A_S = 1.5$ days and $T_S = 47$ days for Japanese stocks. We should emphasize that these numbers (including the previously quoted U.S. results) carry large uncertainties as they vary significantly with the time period and averaging procedure. As a direct test of the retarded model, we studied the correlation between $\delta x_i(t)$ and $(\delta S_i/S_i^R)^2(t + \tau)$. We find this to be zero within error bars, which should be expected if most of the leverage correlations indeed come from a simple retardation effect. Another test of the retarded model when the volatility is fluctuating is to study the residual variance of the local squared volatility $(\delta S_i/S_i^R)^2$ as a function of the horizon T of the averaging kernel. The above value of T_S is found to correspond to a minimum of this quantity.

We therefore conclude that the leverage effect for stocks might not have a very deep economical significance, at variance with some recent claims [14–16], but can be assigned to a simple "retarded" effect, where the change of prices is calibrated not on the instantaneous value of the price but on a moving average of the price. Figure 2, on the other hand, clearly shows that (i) the "leverage effect" for indices is much stronger than that appearing for individual stocks, but (ii) tends to decay to zero much faster with the lag τ. This is at first sight puzzling, since the stock index is, by definition, nothing more than an average over stocks. So one can wonder why the strong index leverage effect does not show up in stocks and conversely why the long time scale seen in stocks disappears from the index. However, this is only an apparent contradiction, as can be seen by studying the (oversimplified [19,20]) one-factor model, where one assumes the following decomposition:

$$\delta S_i(t) = S_i^R(t)[\beta_i\phi(t) + \epsilon_i(t)], \tag{7}$$

where $\phi(t)$ is the return factor common to all the stocks, β_i are some time independent coefficients normalized such that $\sum_i \beta_i = N$, and ϵ_i are the so-called idiosyncrasies, uncorrelated from stock to stock and from the common factor ϕ. The market index $I(t)$ is defined as a certain weighted average of the stocks: $I(t) = \sum_{i=1}^N w_i S_i(t)$, where w_i are certain weights, of order $1/N$. From linearity, one finds the same relation between the retarded quantities. Neglecting terms of order $1/\sqrt{N}$, one therefore finds that $\delta I(t) = I^R(t)\phi(t)$. Now, let us assume that there exists an index specific effect, resulting from an increase of activity when the market as a whole goes down. Downward moves of indices are indeed conspicuous (they often make the headline news) and incite nervous investors to sell their stocks and bargain hunters to buy massively (the leverage effect on indices is even noticeable on intraday data). We define the index specific leverage correlation $\gamma(\tau)$ as

$$\langle\phi^2(t + \tau)\phi(t)\rangle = -\gamma(\tau)\sigma_I^4, \tag{8}$$

where $\sigma_I \equiv \sqrt{\langle\phi^2(t)\rangle}$ is the market volatility. At variance with individual stocks, we do not have a quantitative model for $\gamma(\tau)$. It should be thought of as an empirically measured quantity whose normalization is arbitrary; we choose the factor σ_I^4 for later convenience. Neglecting terms of order $1/\sqrt{N}$, and all mixed effects, we find, for the normalized leverage correlation function for indices,

$$\mathcal{L}_I(\tau) = -2\overline{\mathcal{K}}(\tau) - \gamma(\tau). \tag{9}$$

Therefore, one explicitly sees that the slowly decaying part $\overline{\mathcal{K}}(\tau)$ should in fact also appear in $\mathcal{L}_I(\tau)$. However, the amplitude of this retarded correlation ($= 2$) is only 10% of the observed correlation $[\mathcal{L}_I(0) = 18]$; hence $\gamma(0) \sim 16$. We fitted the observed correlation for indices by a sum of two exponentials, with only the parameters of the "fast" one left free, the slow one being fixed by fitting individual stocks. The resulting fit (not shown) was not significantly different from the single exponential fit of Fig. 2. Given the amount of noise in the data, it is difficult to prove or disprove the presence of the slowly decaying correlation. Nevertheless, we argue that it should be present for reasons of consistency between the index and its constituents. Conversely, let us estimate the contribution of $\gamma(\tau)$ to the individual stock leverage effect. A simple computation gives, to lowest order in η,

$$\mathcal{L}_i(\tau) = -2\overline{\mathcal{K}}(\tau) - \beta_i^3\left(\frac{\sigma_I}{\sigma_i}\right)^4\gamma(\tau). \tag{10}$$

Since the market volatility σ_I is a factor of 2 smaller than the volatility of individual stocks σ_i [19], the coefficient in front of $\gamma(\tau)$ is of order $1/16$. So, even though $\gamma(0)$ is 8 times larger than $2\overline{\mathcal{K}}(0)$, the influence of the market leverage effect on individual stocks is effectively suppressed due to a relatively large ratio between the stock volatility and the market volatility, and by the fast decay of $\gamma(\tau)$. Again, due to the amount of noise in the data, it is difficult to confirm directly the presence of the $\gamma(\tau)$ contribution in $\mathcal{L}_i(\tau)$. However, Fig. 1 suggests that a second, rapidly decaying contribution indeed appears for small τ's. Therefore, although oversimplified, the one-factor model allows us to understand why the stock and index leverage effects do not strongly interfere even if the index is nothing more than an average over all stocks.

In summary, the most important result of this Letter is the fact that the volatility asymmetry on equity markets can be rationalized in terms of a retarded model, which assumes that the reference price used to set the scale for price updates is *not* the instantaneous price but rather a moving average of the price over the past few months. This interpretation, supported by the data on U.S., European, and Japanese stocks, appears to us rather likely, and defines an interesting class of stochastic processes (first advocated in [5]) intermediate between purely additive (valid on short time scales) and purely multiplicative (relevant for long time scales). Several economical interpretations of this effect have been proposed [14,15]. One is the volatility feedback effect [16], where an anticipated increase of future volatility by market operators triggers sell orders, which therefore decrease the price. However, data from option markets (providing the best available volatility forecasts) clearly indicate that volatility forecasts are correlated with observed past volatility [5]. Therefore, the volatility feedback effect should also generate a negative correlation between past volatilities and future prices, in contradiction with our observations. Finally, the parameters of these models and of other asymmetric GARCH models must be tuned to reproduce the order of magnitude of the correlation function that our model naturally predicts. For stock indices the retarded interpretation breaks down and a specific "risk aversion" phenomenon seems to be responsible for the enhanced observed negative correlation between volatility and returns (and in turn to the strong skews observed on index option smiles). Interestingly, this effect appears to decay over a few days, in contrast with the volatility-volatility correlation which extends over several months or years.

We thank M. Meyer, P. Cizeau, E. Bacry, and J. P. Fouque for many interesting discussions. We also thank J. Boersma and Ph. Seager for some valuable comments and suggestions.

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