# Collapse in Systems with Attractive Nonintegrable Potentials

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Collapse, or a gravitational-like phase transition, is found in microcanonical ensembles of particles with an attractive $1/r^\alpha$ potential not only for $\alpha = 1$ but for all $0 < \alpha < 3$. The phase behavior of the system is complex: If an effective sufficiently short-range cutoff is applied, the density of the collapsed phase is finite everywhere; if not, the collapse results in a density singularity. Also, with increasing effective cutoff range, the gravitational phase transition will cross over to a normal first-order phase transition.

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It is known that particle systems with purely attractive gravitational $1/r$ interactions exhibit collapse, sometimes called a zero-order phase transition (cf. Fig. 1 at $e_c$). When the energy in the microcanonical ensemble (ME) or the temperature in the canonical ensemble (CE) drops below a certain critical value $e_c$ or $T_c$, respectively, the corresponding thermodynamic potentials (entropy in the ME or free energy in the CE) undergo a discontinuous jump [1–3]. If no short-range cutoff is introduced, then the discontinuous jump is infinite and the entropy and free energy go to $+\infty$ and $-\infty$, respectively. This makes all normal (noncollapsed) states of the self-attractive system metastable with respect to such a collapse; the collapse energy $e_c$ is in fact an energy below which the metastable states cease to exist.

If, on the other hand, some form of short-range cutoff is introduced, the entropy and free energy jumps are finite. In this case, as a result of the collapse, the system goes into a nonsingular state with a dense core, the precise nature of which depends on the details of the short-range behavior of the potential. Then only the normal states which are in some interval of energies above the collapse point are metastable with respect to such a transition (see Fig. 1). There is an energy $e^*$ for which both collapsed and normal systems have the same entropy (cf. $e_1^*$ and $e_2^*$ in Fig. 1); above this energy the collapsed state becomes metastable and at some higher energy it ceases to exist [4,5]. It is possible to regard the energy $e^*$ as that where a true phase transition occurs. When the effective cutoff vanishes, such phase transition energy $e^*$ increases to infinity. Therefore without a cutoff all the finite energy states are metastable [6]; however, the value of $e^*$ is highly sensitive to the details of the short-range cutoff. On the contrary, the collapse energy $e_c$ depends on the long-range part of the interparticle interactions and is almost unaffected by a cutoff, provided that it is sufficiently short range.

While rather elaborate studies of gravitational collapse have usually been motivated by cosmological applications and were performed solely for an $1/r$ potential, a natural question is as follows: What happens if the particles interact via an attractive $1/r^\alpha$ potential with arbitrary $\alpha$? The most common example of a potential with such power-law dependence is probably the dipole-monopole ($\alpha = 2$) interaction. It has been noticed before that, in systems with nonintegrable interactions, i.e., when $\alpha$ is less than the dimensionality of the space, first-order phase transitions differ in the ME and in the CE, even for $N \to \infty$ [7]. However, in the examples considered in the literature, the potential energy was always bounded from below (usually by putting the system on a lattice), allowing only for normal first-order phase transitions and excluding any singular collapse.

In this Letter, we report on studies of collapse in self-attracting systems, similar to the gravitational Hamiltonian particle systems, but with a general $1/r^\alpha$ potential. Below, we consider three-dimensional systems as being the most common, yet the results can be easily generalized to

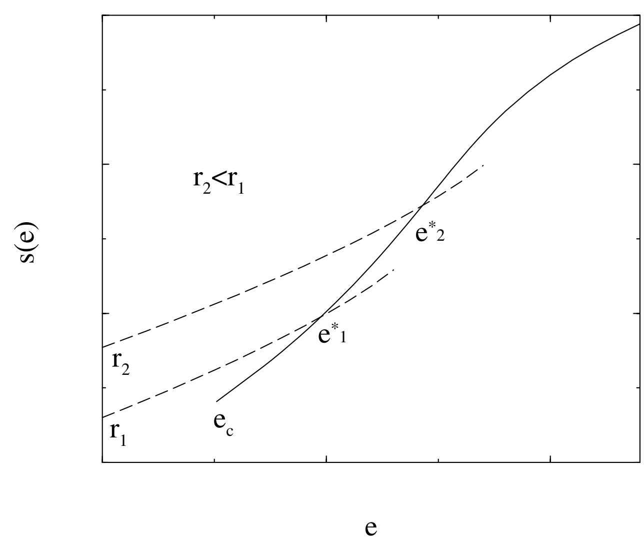


FIG. 1. Sketch of an entropy vs energy plot for a system with gravitational-like collapse. The entropy of the normal (noncollapsed) state is shown by a solid line; the entropies of the two collapsed states for different cutoff ratio $r_1$ and $r_2$, $r_2 < r_1$, are shown by dashed lines. The entropies of the two collapsed states intersect the entropy of the normal state at energies $e_1^*$ and $e_2^*$.

  

arbitrary dimensionality. We will consider systems only in the ME, since it generally allows one to obtain more information about phase transitions than the CE [7–9].

The entropy per particle $s(\epsilon) = S(\epsilon)/N$ in the mean-field (saddle-point) approximation can be expressed as (see [2,10])

$$s(\epsilon) = \frac{3}{2}\ln\left[\epsilon + \frac{1}{2}\iint \frac{\rho(\vec{r}_1)\rho(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^\alpha}\, d\vec{r}_1\, d\vec{r}_2\right] - \int \rho(\vec{r})\ln[\rho(\vec{r})]\, d\vec{r}\,. \quad (1)$$

where $\rho(\vec{r})$ is a solution of the following integral equation:

$$\rho(\vec{r}) = \rho_0 \exp\left[\beta_s \int \frac{\rho(\vec{r}_1)}{|\vec{r}_1 - \vec{r}|^\alpha}\, d\vec{r}_1\right],$$
$$\beta_s = \frac{3}{2}\left[\epsilon + \frac{1}{2}\iint \frac{\rho(\vec{r}_1)\rho(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^\alpha}\, d\vec{r}_1\, d\vec{r}_2\right]^{-1}, \quad (2)$$
$$\rho_0 = \left\{\int \exp\left[\int \frac{\beta_s\rho(\vec{r}_1)}{|\vec{r}_1 - \vec{r}_2|^\alpha}\, d\vec{r}_1\right] d\vec{r}_2^{\,-1}\right\}^{-1}.$$

Here $\epsilon \equiv ER^\alpha/GN^2$ is a dimensionless energy, $R$ is the radius of the confining spherical container, $N$ is the number of particles interacting via a $-G/|\vec{r}_i - \vec{r}_j|^\alpha$ pair potential, and the integrations run over the 3D sphere of radius one. Here we consider only $0 < \alpha < 3$, for which the potential $1/r^\alpha$ is often called nonintegrable, since the integral $\int d^3r/r^\alpha$ diverges at its upper limit. As the potential becomes integrable ($\alpha > 3$), the continuum approach used here becomes inapplicable because the short-range density fluctuations, which the continuous approach cannot account for, become dominant over the long-range effects. Formally, the short-range nature of the behavior of the systems for $\alpha > 3$ manifests itself as the divergence of the integral $\int d^3r/r^\alpha$ at its lower limit.

If $\beta_s$ is considered as an independent parameter rather than a factor depending on $\epsilon$ and $\rho(\vec{r})$, Eq. (2) is often called a generalized Poisson-Boltzmann-Emden equation [11]. Very little is known about this equation even in the gravitational $\alpha = 1$ case. The only exactly known, so-called "singular," solution is for $\beta_s = 2$ [11] and has the form $\rho_{\rm sing}(r) = (4\pi r^2)^{-1}$, which leads to $\epsilon = -1/4$ and $s = \ln(\sqrt{2}\,\pi) - 2$.

To solve Eq. (2) numerically for general $\alpha$ and $\epsilon$, we use a simple iterative method. We put a trial density profile $\rho_0(r)$ into (2), calculate $\beta_s$ and obtain a new density profile $\rho_1(r)$, and repeat this procedure iteratively. In other words, a nonlinear map,

$$\rho_{i+1}(r) = F_\epsilon[\rho_i(\cdot), r], \quad (3)$$

is introduced, with a functional $F_\epsilon[\rho_i(\cdot), r]$ defined by (2). Performing a local stability analysis, we show [10], that the convergence of the map (3) to a certain $\rho(r)$ is a sufficient condition for $\rho(r)$ to be a stable or metastable thermodynamic state. However, to make it a necessary condition as well, i.e., to make the iterative method convergent for all thermodynamically stable or metastable states, we have to introduce a map with a variable "step,"

$$\rho_{i+1}(r) = \sigma F[\rho_i(\cdot), \epsilon] + (1 - \sigma)\rho_i(r)\,, \quad (4)$$

where $0 < \sigma \leq 1$ is the step size parameter. Choosing $\sigma$ sufficiently small (as small as $\sim 10^{-2}$–$10^{-3}$) , we were able to make the algorithm convergent for all density profiles that maximize entropy (1), i.e., are thermodynamically stable or metastable states. Once a sufficient convergence of the iterations (4) has been achieved,

$$4\pi \int_0^1 |\rho_{i+1}(r) - \rho_i(r)| r^2\, dr < \delta \ll 1\,, \quad (5)$$

the entropy is calculated with (1).

The main result that can be derived from the numerical analysis is the following: For *all* $0 < \alpha < 3$, as for $\alpha = 1$, there is a certain energy $\epsilon_c(\alpha)$ below which the system collapses and the entropy exhibits a discontinuous jump. The results for $\epsilon_c(\alpha)$ are presented in Fig. 2.

To verify our calculations of $\epsilon_c(\alpha)$, we compare our result for $\epsilon_c(\alpha = 1)$ with the existing data obtained by other methods. Our number, $\epsilon_c(\alpha = 1) = -0.3346$, is consistent with $\epsilon_c(\alpha = 1) = -0.335$, quoted in [3,4].

To get more insight, let us consider in more detail a system with $\alpha = 1/2$. Plots of the entropy $s(\epsilon)$ and the inverse temperature $\beta_s(e) = ds(\epsilon)/d\epsilon$ of this system are presented in Fig. 3.

As we go down along the energy axis $\epsilon$, the entropy decreases, passing through an inflection point $\epsilon_i$ where $\beta$ reaches its maximum $\beta_m$. For energies below this inflection point, the system has a negative specific heat $[d^2s(\epsilon)/d\epsilon^2 > 0]$ and is therefore unstable in the CE. As we pass through the $\epsilon_i$ point and continue decreasing the energy, the convergence of (3) becomes slower and slower, and at the point $\epsilon_c$ the iterations start to diverge. It is straightforward to show for all $0 < \alpha < 3$ (see, e.g., [11] for $\alpha = 1$) that the entropy is unbounded from above with

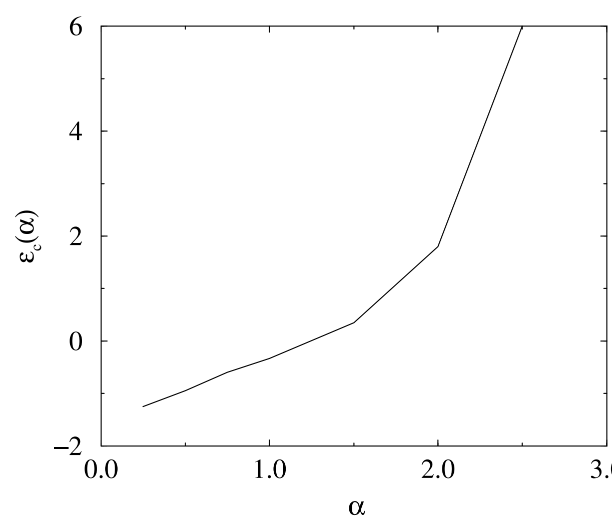


FIG. 2. Plot of collapse energy $\epsilon_c(\alpha)$ vs potential exponent $\alpha$.

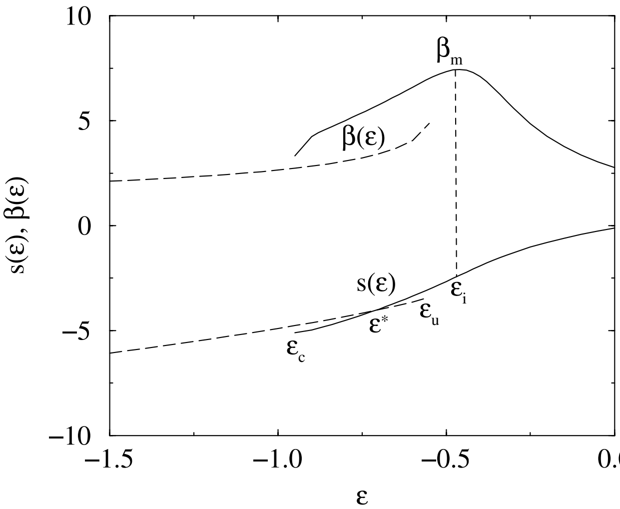


FIG. 3. Plots of entropy $s(\epsilon)$ and entropy derivative $\beta_s(\epsilon)$ for noncollapsed states (solid lines) and collapsed states (dashed lines) for $\alpha = 1/2$. The radius of excluded central volume $r_0 = 5 \times 10^{-4}$. The points $\epsilon_c, \epsilon^*, \epsilon_u$, and $\epsilon_i$ are defined in the text.

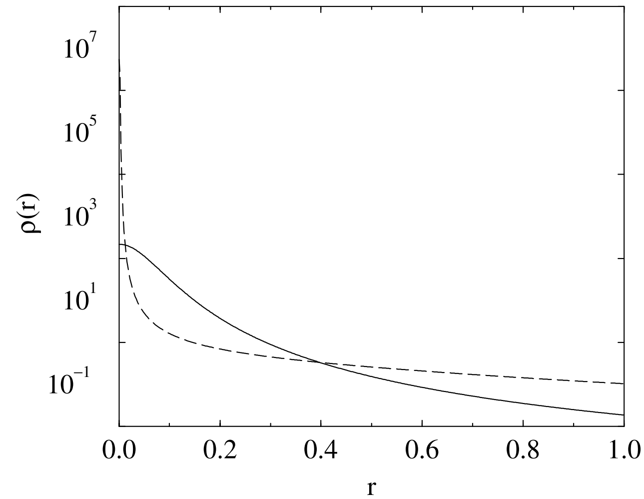


FIG. 4. Density profiles $\rho(r)$ for $\alpha = 1/2$ for noncollapsed (solid line) and collapsed (dashed line) phases for the energy $\epsilon^* = -0.708$, when entropies of both phases are the same. The radius of the excluded central volume is $r_0 = 5 \times 10^{-4}$.

respect to uniform squeezing of all the matter into a sphere with a radius going to zero. Hence, if no short-range cutoff is present, it is reasonable to assume that the entropy discontinuity at $\epsilon_c(\alpha)$ is infinite.

If some form of a short-range cutoff is introduced, the entropy discontinuity may become finite. To investigate this we tried two approaches. One, suggested in [4], is to place a small spherical excluded volume with a radius $r_0$ in the center of the system, or, in other words, to replace a spherical container with a spherical shell container. The other approach is to replace the original "bare" potential $1/r^\alpha$ with a "soft" potential of the form $1/(r^2 + r_0^2)^{2\alpha}$. For a reasonably small short-range cutoff ($r_0 \sim 10^{-3}$ for small $\alpha$, $r_0 \sim 10^{-2}$ for $\alpha \simeq 3$ for both approaches) the behavior of the noncollapsed system is virtually unaffected. A typical density profile in the collapsed phase exhibits a much higher concentration around the origin than the normal (noncollapsed) phase; plots of density profiles for $\alpha = 1/2$ are presented in Fig. 4.

A collapsed phase exists not only for $\epsilon < \epsilon_c$, but for $\epsilon > \epsilon_c$ as well. In fact, this phase is globally stable in the range of energies where its entropy is higher than that of the normal phase, i.e., when $\epsilon < \epsilon^*$. For $\epsilon > \epsilon^*$, the collapsed phase is metastable and above some energy $\epsilon_u$ becomes unstable even locally (see Fig. 3).

Finally we return to the exact $\rho_{\text{sing}}(r) = (4\pi r^2)^{-1}$ solution which exists for $\epsilon = -1/4$ and $\alpha = 1$. Our attempts to approach this solution by the numerical iterative methods (3) and (4) failed. In fact, even after substituting the $\rho_{\text{sing}}(r)$ into (4) as an initial approximation $\rho_0(r)$, the iterative solution of (4) evolved either to a normal or to a collapsed solution depending on the value of the step $\sigma$. We calculated the entropies for the three solutions that exist at $\epsilon = -1/4$: a normal $s_n$, a collapsed $s_c$, and a $s_{\text{sing}}$ for $\rho_{\text{sing}}$. It turns out that $s_{\text{sing}} < \min\{s_n, s_c\}$, which, together with the evidence obtained from the iterative procedures mentioned above, strongly suggests that, in the space of solutions (or fixed points) of (4), both normal and collapsed $\rho(x)$ are at least locally stable (attractive), while $r_{\text{sin}}$ is unstable (repulsive).

A very important question is that of the order of the gravitationlike phase transition. Here we have to distinguish between the collapse itself, which happens at $\epsilon_c$, and the "true" phase transition which happens at the energy $\epsilon^*$ where the entropies of noncollapsed and collapsed states are equal (see Fig. 3). Since the entropy at the collapse point $\epsilon_c$ exhibits a discontinuous jump, the collapse is often called a zero-order phase transition [2]. However, the collapse is not a phase transition in the normal sense since it converts a metastable state into a stable one, which can be either singular or finite, depending on the presence of a short-range cutoff.

On the other hand, the true phase transition between stable phases, which happens at $e^*$, is sometimes referred to as a "gravitational first-order phase transition" [5]. Its distinct features include an inability of the two phases (noncollapsed and collapsed) to coexist as well as a discontinuous $\beta(\epsilon)$, i.e., temperature [5]. Yet in a "normal" ME first-order phase transition in a long-range interacting system (such as a mean-field Potts model), $\beta(\epsilon)$ remains continuous and smooth, but exhibits nonmonotonic behavior: The interval of energies where phases coexist includes an interval where $d\beta(\epsilon)/d\epsilon$ is positive and the specific heat is negative (see Fig. 5) [7–9]. Hence there is an intrinsic difference between the normal and the gravitational first-order phase transitions. Remarkably, normal first-order phase transitions are found to replace gravitational first-order phase transitions, which occur in the self-attracting systems considered here, if the short-range

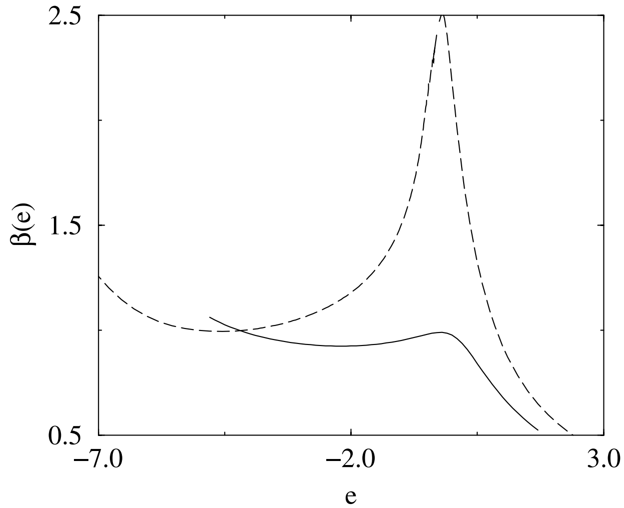


FIG. 5. Entropy derivative $\beta_s(\epsilon) = ds(\epsilon)/d\epsilon$ vs energy $\epsilon$ plot for $\alpha = 2$ and central core radius 0.5 (solid line), and $\alpha = 1$ and soft potential radius 0.05 (dashed line).

cutoff is sufficiently increased. As was noted in [5] for $\alpha = 1$, there is a critical excluded volume radius $r_c$ above which there is no discontinuity in the entropy vs energy plot. We observed that this trend is generic for all $0 < \alpha < 3$ and holds for both excluded volume and soft potential cutoffs. The critical cutoff radius $r_c(\alpha)$ increases with increasing $\alpha$, roughly varying in value from below $10^{-3}$ for $\alpha = 1/4$, to above $10^{-1}$ for $\alpha = 5/2$, respectively. For a system with a cutoff radius larger than $r_c(\alpha)$, the entropy vs energy plot is continuous and exhibits all characteristics of a normal first-order phase transition in the ME [7,9]: a convex dip and associated with it an interval of energies, where $d^2s(\epsilon)/d\epsilon^2$ is positive and the heat capacity is negative (Fig. 5).

In summary, in this paper we revealed that a collapse and, associated with it, a discontinuity in the microcanonical ensemble entropy exist not only in self-gravitating systems, but in all ensembles of particles with general $1/r^\alpha$, $0 < \alpha < 3$ attractive potential. This discontinuity was an infinite jump if no short-range cutoff was present. A carefully introduced short-range cutoff leaves the properties of the noncollapsed system virtually unaffected, but makes the entropy jump finite and allows one to observe the collapsed phase. The stability of a solution $\rho(r)$ of the integral equation (4) is the necessary and sufficient condition for the density profile $\rho(r)$ to make the entropy a maximum and therefore to represent either a stable or a metastable state. Furthermore, as the short-range cutoff of the potential increases, the collapse disappears, and the gravitational-like first-order phase transition becomes a normal first-order phase transition, Apart from astrophysical applications, our results may well be important in condensed matter physics where nonintegrable potentials are quite common.

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