Proof of Factorization for $B \to D\pi$

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We prove that the matrix elements of four fermion operators mediating the decays $\bar{B}^0 \to D^+\pi^-$ and $B^- \to D^0\pi^-$ factor into the product of a form factor describing the $B \to D$ transition and a convolution of a short distance coefficient with the nonperturbative pion light-cone wave function. This is shown to all orders in α_s, with corrections suppressed by factors of $1/m_b$, $1/m_c$, and $1/E_\pi$. It is not necessary to assume that the pion state is dominated by the $q\bar{q}$ Fock state.

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1. Introduction.— In this Letter we present an all orders proof of factorization for $\bar{B}^0 \to D^+\pi^-$ and $B^- \to D^0\pi^-$, in the limit where the heavy quark masses approach infinity. To be explicit, we distinguish three types of factorization. In this Letter we prove the generalized factorization of matrix elements of four quark operators into a form factor describing the $B \to D$ transition and a convolution of a short distance coefficient with the pion wave function [1,2]. A second type of factorization, between hard and infrared scales, is related to defining the correct effective theory as explained below. Finally, a third type is factorization theorems between soft and collinear degrees of freedom [3] which are also discussed.

The $B \to D\pi$ decays are mediated by the "full theory" weak Hamiltonian at a scale $\mu_0 \sim m_b$

$$\mathcal{H}_W = \frac{4G_F}{\sqrt{2}} V_{ud}^* V_{cb} [C_0^F(\mu_0) O_0(\mu_0) + C_8^F(\mu_0) O_8(\mu_0)].$$
(1)

The operators are

$$O_0 = [\bar{c}\gamma^\mu P_L b][\bar{d}\gamma_\mu P_L u],$$
$$O_8 = [\bar{c}\gamma^\mu P_L T^a b][\bar{d}\gamma_\mu P_L T^a u],$$
(2)

with $P_L = (1 - \gamma^5)/2$. Generalized factorization [1,2] says that for $B \to D\pi$ decays where the light degrees of freedom in the B can end up in the D, the matrix elements of $O_{0,8}$ can be factored according to

$$\langle D\pi|O|B\rangle = N F^{B \to D}(0) \int_0^1 dx\, T(x,\mu)\phi_\pi(x,\mu),$$
(3)

where $F^{B \to D}(0)$ is a $B \to D$ form factor at $q^2 = 0$, $N = im_B E_\pi f_\pi/2$, and $\phi_\pi(x,\mu)$ is the nonperturbative light-cone pion wave function [4]. Finally, $T(x,\mu)$ is a computable short distance coefficient and is a function of the renormalization scale μ, the matching scale μ_0, as well as x and $z = m_c/m_b$. The earliest form of (3) is so-called naive factorization where one sets $T(x,m_b) = 1$, dropping $\alpha_s(m_b)$ corrections. The first argument for naive factorization was based on the idea of color transparency [5]. The physical picture is simply that long wavelength gluons cannot resolve the existence of individual colored objects in the fast moving pion, and thus decouple. The first

attempt to prove naive factorization was by Dugan and Grinstein in the context of a large energy effective theory (LEET) [6]. This theory contains soft gluons coupling to collinear quarks, but in $n \cdot A = 0$ gauge this coupling vanishes. Thus, no soft gluons can connect the heavy quarks to the light fermions. However, LEET omits collinear gluons. In [7] the generalized factorization formula in (3) was shown to be valid at two loops in perturbation theory, including collinear gluon interactions. This two-loop convolution was reproduced in a soft-collinear effective theory in [8]. In this paper this theory combined with heavy quark effective theory (HQET) is used to extend the proof of factorization to all orders in perturbation theory.

2. Effective theory.— The soft-collinear effective theory [8–10] describes processes with final state particles having energy much larger than their mass. For $B \to D\pi$ the pion has large energy, and we take the limit $Q \gg \Lambda_{QCD}$ where Q is E_π, m_b, or m_c. Momenta $k^\mu \gtrsim Q$ are integrated out and contribute to Wilson coefficients in the effective theory. The remaining infrared physics can be described by including all on-shell degrees of freedom whose momenta are set by the scales in the process. The heavy mesons can be described by heavy HQET quarks (h_v), soft quarks (q_s), and soft gluons (A_s^μ), all with momenta of order $Q\lambda$, where $\lambda \sim \Lambda_{QCD}/Q \ll 1$. The fast moving pion contains collinear quarks ($\xi_{n,p}$) and collinear gluons ($A_{n,p}^\mu$), with momenta scaling as $(p^+, p^-, p^\perp) \sim Q(\lambda^2, 1, \lambda)$. All four components of $A_{n,q}^\mu$ give order λ^0 interactions with collinear quarks and are responsible for binding the pion constituents. In addition, ultrasoft gluons (A_{us}^μ) with momenta $k_{us}^\mu \sim Q\lambda^2$ can be emitted by a collinear quark without changing the scaling of its momenta (i.e., taking it off its mass shell). The HQET fields are labeled by the heavy quark velocity v, while collinear quarks and gluons are labeled by their light-cone direction n and the large part of their momentum. The same modes for gluons and quarks also appear in the method of regions [7,11].

To simplify the power counting, fields are rescaled by powers of λ to make all kinetic terms $\mathcal{O}(\lambda^0)$. This gives $h_v \sim q_s \sim \lambda^{3/2}$, $(A_{n,q}^+, A_{n,q}^-, A_{n,q}^\perp) \sim (\lambda^2, 1, \lambda)$, $\xi_{n,p} \sim \lambda$, $A_s^\mu \sim \lambda$, $A_{us}^\mu \sim \lambda^2$, and $q_{us} \sim \lambda^3$. Using topological identities the power of λ for an arbitrary diagram can then

be determined entirely from the interaction vertices, and only $\mathcal{O}(\lambda^0)$ Feynman rules are required. For $B \to D\pi$ a graph is $\mathcal{O}(\lambda^\delta)$ with $\delta + 1 = \sum_k [(k - 8)V_k^{us} + (k - 4)(V_k^s + V_k^c + V_k^{cs})]$. V_k^i counts interaction field operators of type i with scaling λ^k (V^{cs} are mixed collinear-soft vertices). For example, a single $\bar{h}_{v'} h_v \bar{\xi}_{n,p'} \xi_{n,p}$ is $V_5^{sc} = 1$, so $\delta = 0$. The couplings of soft gluons to heavy quarks are identical to HQET, and those of soft quarks and gluons are simply given by QCD.

3. Preliminaries.—We wish to show that at leading order in λ the effective theory Feynman rules leave diagrams only of the form shown in Fig. 1, so that no nonfactorizable infrared contributions occur. This picture illustrates how, even in the presence of arbitrary hard interactions, soft gluons decouple from the pion and collinear gluons couple to the hard vertex [which gives rise to the convolution in (3)]. Arguments for the former are fairly standard but are given for our case. The convolution is more interesting. We begin by showing that in the absence of hard gluons, collinear gluons completely decouple from the B and D (naive factorization). We then prove (3) (generalized factorization) by using the fact that the form of operators induced by integrating out hard gluons are constrained by a symmetry [8,10].

We will assume that the tail end of wave functions are suppressed by λ^a with $a > 0$. For the pion, these configurations have a single valence quark carrying off most of the energy, and for the B and D they contain a spectator with momentum $\gg \Lambda_{QCD}$. These assumptions can be used
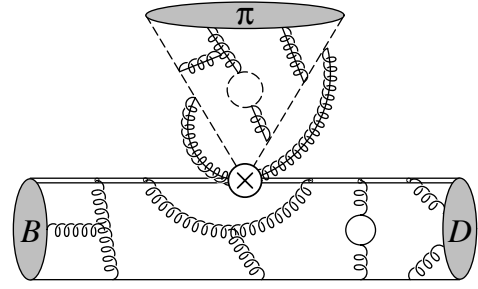

FIG. 1.   How the factorization of modes takes place.

to show the power suppression of annihilation and hard spectator contributions, respectively [7].

4. Naive factorization.—To build some intuition, we begin by neglecting all hard matching corrections proportional to $\alpha_s(Q)$, but work to all orders in the couplings of the effective theory gluons. In this case we show that the sum of all diagrams with gluons connecting quarks in the heavy mesons to those in the pion is zero.

Collinear gluons cannot couple to the heavy quarks since an HQET quark cannot emit or absorb a collinear gluon and stay near its mass shell [9]. Instead, the coupling of collinear gluons to heavy quarks introduces nonlocal operators, which *a priori* can still spoil factorization. To match onto these operators at tree level we follow [10]. An infinite number of $A_{n,q}$ gluons contribute to the matching onto any operator with a heavy quark as in Fig. 2. Since $A_{n,q}^\mu = \bar{n} \cdot A_{n,q} n^\mu/2 + \mathcal{O}(\lambda)$, only the $\bar{n} \cdot A_{n,q}$ gluons appear at $\mathcal{O}(\lambda^0)$. For one such gluon

$$-g\Gamma \frac{m_b \not{v} + \not{q}_1 + m_b}{(m_b v + q_1)^2 - m_b^2} \frac{\not{n}}{2} (\bar{n} \cdot A)b = -g \frac{(\bar{n} \cdot A_{n,q_1})}{\bar{n} \cdot q_1} \Gamma \frac{(1 + \not{v})\not{n}}{2v \cdot n} h_v = -g \frac{(\bar{n} \cdot A_{n,q_1})}{\bar{n} \cdot q_1} \Gamma h_v, \qquad (4)$$

using $\not{v} h_v = h_v$. It is important to note that (4) is independent of the value of $v \cdot n$, and thus independent of the heavy quark velocity v. This matching can be extended to include an arbitrary number of collinear gluons [10]

$$\sum_{m,\text{perms}} \frac{(-g)^m}{m!} \frac{(\bar{n} \cdot A_{n,q_m}) \cdots (\bar{n} \cdot A_{n,q_1})}{(\bar{n} \cdot q_1) \cdots (\sum_{i=1}^m \bar{n} \cdot q_i)} \Gamma h_v \equiv W\Gamma h_v. \qquad (5)$$

With these definitions, the effective Hamiltonian below $\mu_0 \sim Q$ matches at tree level onto the operators

$$Q_{0,\text{tree}}^{1,5} = [\bar{h}_{v'}^{(c)} \Gamma_h^{1,5} h_v^{(b)}][\bar{\xi}_{n,p'}^{(d)} \Gamma_\ell \xi_{n,p}^{(u)}],$$
$$Q_{8,\text{tree}}^{1,5} = [\bar{h}_{v'}^{(c)} \Gamma_h^{1,5} (W^\dagger T^A W) h_v^{(b)}][\bar{\xi}_{n,p'}^{(d)} \Gamma_\ell T^A \xi_{n,p}^{(u)}], \qquad (6)$$

where $\Gamma_h^1 = \not{n}/2$, $\Gamma_h^5 = \not{n}\gamma_5/2$, and $\Gamma_\ell = \bar{\not{n}}(1 - \gamma_5)/4$. For $Q_{0,\text{tree}}^i$ we have used $W^\dagger W = 1$, which encodes the important observation that collinear gluon interactions from the b and c quarks cancel identically to all orders for the color singlet operators. It is not possible to add additional fields to (6), such as a soft gluon, without increasing the power of λ. A collinear gluon could also interact with the spectator quark in the B to change it into a collinear quark. However, this interaction does not occur at $\mathcal{O}(\lambda^0)$ because $\not{n}\xi_{n,p} = 0$.

We defer to the next section the proof that only soft gluons exchanged between the partons in the B and D contribute, as in Fig. 1. Assuming this, naive factorization is obtained by showing that $\langle D\pi | Q_{8,\text{tree}}^i | B \rangle$ vanishes, while $\langle Q_{0,\text{tree}}^i \rangle$ factors into the product of matrix elements of two currents. Let M denote an arbitrary color structure associated with soft modes exchanged between color singlet B and D states. Since all adjoint indices in M are contracted, the lower color trace in $\langle Q_{8,\text{tree}}^i \rangle$ is

$$\text{Tr}[MW^\dagger T^A W] = M \text{Tr}[W^\dagger T^A W] \propto \text{Tr}[T^A] = 0. \quad (7)$$

By parity $\langle Q_{0,\text{tree}}^5 \rangle$ vanishes. Finally, $Q_{0,\text{tree}}^i$ contains no collinear gluons, so *no* gluons connect the soft and
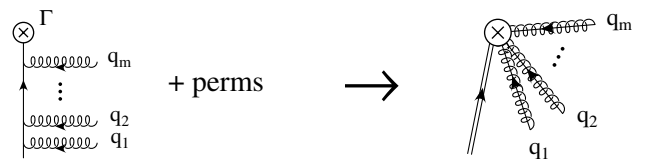

FIG. 2.   Matching for the order λ^0 Feynman rule with a heavy quark and m collinear gluons.

collinear partons at $\mathcal{O}(\lambda^0)$. Thus, $\langle Q_{\mathbf{0},\text{tree}}^i\rangle$ factors

$$\langle D_{v'}\pi_n|Q_{\mathbf{0},\text{tree}}^1|B_v\rangle = \frac{i}{2}E_\pi f_\pi m_B F^{B\to D}(0) + \dots. \quad (8)$$

Equation (8) is the product of the pion decay constant from $E_\pi f_\pi = \frac{i}{2}\langle\pi_n|\bar{\xi}_{n,p'}\bar{\not{n}}\gamma^5\xi_{n,p}|0\rangle$ with $p_\pi^\mu = E_\pi n^\mu$, and the $B\to D$ form factor $F^{B\to D}(0) = \frac{1}{2}(m_D/m_B)^{1/2}\times(1+m_B/m_D)\xi(v\cdot v')$, where $\xi(v\cdot v')$ is the Isgur-Wise function [12]. The states in (8) are in the effective theory (with relativistic normalization), and the ellipses denote terms suppressed by $1/Q$ or $\alpha_s(Q)$. The result in (8) is exactly the statement of naive factorization.

5. Decoupling of ultrasoft and soft gluons.—By simple power counting the couplings of ultrasoft gluons to heavy quarks and soft modes are suppressed by at least one power of λ. For example, $\bar{h}_v A_{us}^\mu h_v \sim \lambda$, i.e., $V_5^s = 1$. (If ultrasoft heavy quarks are allowed as in [7] then decoupling A_{us}^μ gluons follows the proof for A_s^μ gluons below.) To prove factorization, we therefore need to show that interactions between soft gluons and "collinear" particles with $\bar{n}\cdot p\sim Q$ decouple. For this section only, the name "collinear" will be used to refer to *any* particles with $\bar{n}\cdot p\sim Q$. This includes the degrees of freedom discussed in Sec. 3, as well as off-shell fluctuations with $p^2 < Q^2$ [for example, quarks and gluons with momenta $(k^+, k^-, k^\perp)\sim Q(\lambda, 1, \lambda)$].

The decoupling of soft gluons from collinear particles is a standard part of the proof of QCD factorization theorems for processes such as Drell-Yan [3]. The decoupling depends on only soft $n\cdot A_s$ gluons coupling to collinear fields at $\mathcal{O}(\lambda^0)$, and that the soft k^- and $k^\perp$ momenta drop out of collinear propagators. Applying Ward identities then factors arbitrary soft attachments out of any time ordered product of collinear fields.

The power counting can be used to derive that only $n\cdot A$ soft (ultrasoft) gluons couple to collinear particles at λ^0. (The offshell collinear modes can be included by treating them as auxiliary fields.) At lowest order we find only the Feynman rules shown in Fig. 3. We see immediately that all soft (ultrasoft) gluons couple proportional to n^μ.

In the effective theory the soft k^- and $k^\perp$ momenta drop out of collinear propagators. This occurs due to the large $\bar{n}\cdot p_c$ component for a collinear momentum p_c, so that $(p_c+k)^2 = \bar{n}\cdot p_c\, n\cdot k + \mathcal{O}(\lambda^2)$. For ultrasoft gluons

these momenta drop out using the multipole expansion and equations of motion in the Lagrangian [9].

Now, $n\cdot A_s$ gluons couple to a collinear time ordered product T_c which has dependence only on k^+ momenta, for example, $A_s(k)\cdot T_c = (n\cdot A_s)\bar{n}\cdot T_c(k^+)/2 = (n\cdot A_s)k\cdot T_c/(n\cdot k)$. Thus, QCD Ward identities can be applied. By induction all soft gluons can be decoupled from T_c into eikonal line prefactors [3], $S = P\exp[ig\int dx\, n\cdot A_s(xn^\mu)]$. For the operator $Q_{\mathbf{0}}^i$, unitarity gives $S^\dagger S = 1$ and the soft gluons decouple. For $Q_{\mathbf{8}}^i$ one obtains a color structure $T^a\otimes WS^\dagger T^a SW^\dagger = ST^a S^\dagger\otimes WT^a W^\dagger$ and the vanishing of the octet matrix element in (7) is still obtained.

6. Generalized factorization.—To include arbitrary hard corrections we cannot rely on tree level matching as was done to determine the operators in (6). Since momenta $\gtrsim Q$ are integrated out, the Wilson coefficients in the effective theory are in general arbitrary functions of the large $\bar{n}\cdot p_i$ momenta [10]. In [8] it was pointed out that this functional dependence is greatly restricted by a symmetry induced by collinear gauge transformations. Under this symmetry, $\xi_{n,p}$ and $A_{n,q}^\mu$ transform, but h_v does not since collinear gluons do not couple to nearly on-shell heavy quarks. For $B\to D\pi$ the most general allowed leading order operators are [8]

$$Q_{\mathbf{0}}^j = [\bar{h}_{v'}^{(c)}\Gamma_h^j h_v^{(b)}][\bar{\xi}_{n,p'}^{(d)}WC_{\mathbf{0}}^j(\bar{\mathcal{P}}_+)\Gamma_\ell W^\dagger\xi_{n,p}^{(u)}], \quad (9)$$

$$Q_{\mathbf{8}}^j = [\bar{h}_{v'}^{(c)}S\Gamma_h^j T^a S^\dagger h_v^{(b)}][\bar{\xi}_{n,p'}^{(d)}WC_{\mathbf{8}}^j(\bar{\mathcal{P}}_+)\Gamma_\ell T^a W^\dagger\xi_{n,p}^{(u)}],$$

where $j = 1,5$. Helicity ensures that only Γ_ℓ is needed between the light quarks. The dimensionless Wilson coefficients $C_{\mathbf{a}}^i$ are functions of the renormalization scale μ, as well as m_b, m_c, $v\cdot v'$, and the label operators $\bar{\mathcal{P}}$ and $\bar{\mathcal{P}}^\dagger$ [8]. Since $\bar{\mathcal{P}}$ does not commute with collinear fields, the short distance Wilson coefficient is conveniently included as part of the $Q_{\mathbf{a}}^i$'s. In terms of the label operators

$$W = \left[\sum_{\text{perms}}\exp\left(-g\frac{1}{\bar{\mathcal{P}}}\bar{n}\cdot A_{n,q}\right)\right], \quad (10)$$

and $W^\dagger\xi_{n,p}$ is an invariant under a collinear gauge transformation. The operators $\bar{\mathcal{P}}$ and $\bar{\mathcal{P}}^\dagger$ give the sum of labels on collinear fields to their right and left, respectively, and are described in detail in [8]. For, e.g., if f is some function then $f(\bar{\mathcal{P}})(\bar{\xi}_{n,p'}A_{n,q}^\mu A_{n,r}^\nu\xi_{n,p}) = f(\bar{n}\cdot q + \bar{n}\cdot r + \bar{n}\cdot p - \bar{n}\cdot p')\times(\bar{\xi}_{n,p'}A_{n,q}^\mu A_{n,r}^\nu\xi_{n,p})$. For the $B\to D\pi$
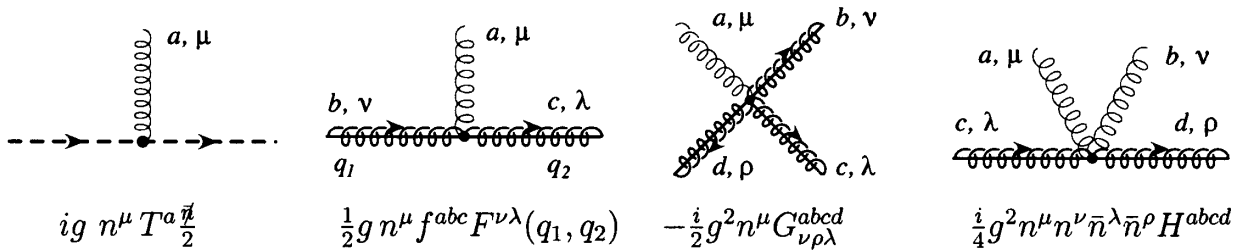

$$ig\, n^\mu T^a\frac{\bar{\not{n}}}{2}\qquad\qquad \frac{1}{2}g\, n^\mu f^{abc}F^{\nu\lambda}(q_1,q_2)\qquad\qquad -\frac{i}{2}g^2 n^\mu G_{\nu\rho\lambda}^{abcd}\qquad\qquad \frac{i}{4}g^2 n^\mu n^\nu\bar{n}^\lambda\bar{n}^\rho H^{abcd}$$

FIG. 3. Order λ^0 Feynman rules for coupling ultrasoft or soft gluons (spring lines) to "collinear" fermions (thick dashed lines) and "collinear" gluons (thick spring + solid lines) (cf. Sec. 5). As an example, coupling an ultrasoft gluon to two collinear gluons in background field Feynman gauge one finds $F^{\nu\lambda}(q_1,q_2) = 2\bar{n}\cdot q_1 g^{\nu\lambda}$, which is $V_4^c = 1$.

matrix element the combination $\bar{\mathcal{P}}^\dagger - \bar{\mathcal{P}}$ behaves like a total derivative, and by momentum conservation gives the total large momentum label of the effective theory state [8], $\bar{\mathcal{P}}^\dagger - \bar{\mathcal{P}} = 2E_\pi$. The dependence on the other linear combination $\bar{\mathcal{P}}_+ = \bar{\mathcal{P}} + \bar{\mathcal{P}}^\dagger$ is displayed explicitly in (9).

If we neglect hard corrections ($C_0^i = 1$) and leave soft-collinear couplings in a Lagrangian, then (9) reduces to (6). This follows from the color identity $W^\dagger T^A W \otimes T^A = T^A \otimes W T^A W^\dagger$, which connects the picture where W is obtained by integrating out off-shell heavy quarks to the picture where W appears by demanding invariance under collinear gauge symmetry in the effective theory.

Up to power corrections the full theory matrix element is $\langle D\pi | C_a^F O_a | B \rangle = \langle D_{v'} \pi_n | Q_b | B_v \rangle$ (summing over $\mathbf{a}, \mathbf{b}$). Therefore, we must simply prove generalized factorization for the effective theory matrix element. For $B \to D\pi$ the same arguments used in Sec. 4 rule out contributions from Q_8^i and Q_0^5. For Q_0^1,

$$\langle D_{v'} \pi_n | Q_0^1 | B_v \rangle = m_B F^{B \to D} \langle \pi_n | \bar{\xi}_{n,p'} \Gamma_\ell W C_0^1(\bar{\mathcal{P}}_+) W^\dagger \xi_{n,p} | 0 \rangle$$

$$= m_B F^{B \to D} \int d\omega \, C_0^1(\omega) \langle \pi_n | \bar{\xi}_{n,p'} \Gamma_\ell W \delta(\omega - \bar{\mathcal{P}}_+) W^\dagger \xi_{n,p} | 0 \rangle. \tag{11}$$

In the first equality we used that collinear gluons do not connect to particles in the heavy meson states, while soft gluons do not connect to those in the pion. The second equality follows trivially, but illustrates how the noncommutative nature of the Wilson coefficients and fields leads to a convolution. In our formulas hard corrections to the $B \to D$ form factor are contained in C_0^1.

Next we show that the matrix element in the last line of (11) is the Fourier transform (FT) of

$$\langle \pi_n^-(p_\pi) | \bar{\xi}_n^{(d)}(y) \slashed{n} \gamma_5 \mathcal{W}(y, -y) \xi_n^{(u)}(-y) | 0 \rangle \equiv -2if_\pi E_\pi \int_0^1 dx \, \phi_\pi(x, \mu) e^{i2yE_\pi(2x-1)}, \tag{12}$$

where the FT of $\xi_n(y)$ is $\xi_{n,p}$, $\mathcal{W}(y, -y)$ is the path ordered eikonal line from positions $-y\bar{n}^\mu$ to $y\bar{n}^\mu$, and $\phi_\pi(x, \mu)$ is the light-cone pion wave function. Since the FT of $[\bar{\xi}_n(y) \mathcal{W}(y, \infty)]$ with respect to R is $\bar{\xi}_{n,p} W \delta_{\bar{\mathcal{P}}^\dagger, R}$, the Fourier transform of (12) is

$$\int \frac{dy}{2\pi} e^{i\omega(-y)} \langle \pi_n | \bar{\xi}_n(y) \Gamma_\ell \mathcal{W}(y, -y) \xi_n(-y) | 0 \rangle = \int \frac{dy}{2\pi} e^{-i\omega y} \langle \bar{\xi}_n(y) \mathcal{W}(y, \infty) \Gamma_\ell \mathcal{W}^\dagger(-y, \infty) \xi_n(-y) \rangle$$

$$= \int \frac{dy}{2\pi} \sum_{R,T} e^{-i\omega y} e^{i(R+T)y} \langle \bar{\xi}_{n,p'} \Gamma_\ell W \delta_{\bar{\mathcal{P}}^\dagger, R} \delta_{\bar{\mathcal{P}}, T} W^\dagger \xi_{n,p} \rangle$$

$$= \sum_{R,T} \delta(\omega - T - R) \langle \bar{\xi}_{n,p'} \Gamma_\ell W \delta_{\bar{\mathcal{P}}^\dagger, R} \delta_{\bar{\mathcal{P}}, T} W^\dagger \xi_{n,p} \rangle$$

$$= \langle \pi_n | \bar{\xi}_{n,p'} \Gamma_\ell W \delta(\omega - \bar{\mathcal{P}}_+) W^\dagger \xi_{n,p} | 0 \rangle. \tag{13}$$

Thus, the convolution in Eq. (11) is

$$\int d\omega \, C_0^1(\mu, \omega) \langle \pi_n | \bar{\xi}_{n,p'} \Gamma_\ell W \delta(\omega - \bar{\mathcal{P}}_+) W^\dagger \xi_{n,p} | 0 \rangle = \frac{i}{2} f_\pi E_\pi \int \frac{dy}{2\pi} d\omega \int_0^1 dx \, e^{iy[2(2x-1)E_\pi - \omega]} C_0^1(\mu, \omega) \phi_\pi(x, \mu)$$

$$= \frac{i}{2} f_\pi E_\pi \int_0^1 dx \, C_0^1(\mu, 2(2x-1)E_\pi) \phi_\pi(x, \mu) = \frac{i}{2} f_\pi E_\pi \int_0^1 dx \, T(x, \mu) \phi_\pi(x, \mu), \tag{14}$$

where $T(x, \mu) \equiv C_0^1(\mu, (4x-2)E_\pi)$.

Combining (11) and (14) we arrive at

$$\langle D_{v'} \pi_n | Q_0^1 | B_v \rangle = N F^{B \to D}(0) \int_0^1 dx \, T(x, \mu) \phi_\pi(x, \mu).$$

This is our final result, and it reproduces the generalized factorization formula in Eq. (3). Note that it was not necessary to set the transverse momenta of partons to zero. It should be fairly obvious that this proof also goes through for other class I decays. Q.E.D.

[1] H. Politzer and M. Wise, Phys. Lett. B **257**, 399 (1991).
[2] M. Beneke *et al.*, Phys. Rev. Lett. **83**, 1914 (1999).
[3] J. Collins, D. Soper, and G. Sterman, Nucl. Phys. **B261**, 104 (1985); **B308**, 833 (1988).
[4] P. Lepage and S. Brodsky, Phys. Rev. D **22**, 2157 (1980).
[5] J. Bjorken, Nucl. Phys. Proc. Suppl. **11**, 325 (1989).
[6] M. Dugan and B. Grinstein, Phys. Lett. B **255**, 583 (1991).
[7] M. Beneke *et al.*, Nucl. Phys. **B591**, 313 (2000).
[8] C. Bauer and I. Stewart, Phys. Lett. B **516**, 134 (2001).
[9] C. Bauer, S. Fleming, and M. Luke, Phys. Rev. D **63**, 014006 (2001).
[10] C. Bauer, S. Fleming, D. Pirjol, and I. Stewart, Phys. Rev. D **63**, 114020 (2001).
[11] V. Smirnov and E. Rakhmetov, Theor. Math. Phys. **120**, 870 (1999); M. Beneke and V. Smirnov, Nucl. Phys. **B522**, 321 (1998).
[12] A. Manohar and M. Wise, *Cambridge Monographs on Particle Physics, Nuclear Physics, and Cosmology* (Cambridge University Press, Cambridge, U.K., 2000), Vol. 10.