An Integrable Shallow Water Equation with Linear and Nonlinear Dispersion

Holger R. Dullin*
Department of Mathematical Sciences, Loughborough University, Loughborough, United Kingdom

Georg A. Gottwald†
Department of Mathematics and Statistics, University of Surrey, Guildford, Surrey, United Kingdom

Darryl D. Holm‡
CNLS and Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
(Received 12 March 2001; published 17 October 2001)

We use asymptotic analysis and a near-identity normal form transformation from water wave theory to derive a 1 + 1 unidirectional nonlinear wave equation that combines the linear dispersion of the Korteweg-deVries (KdV) equation with the nonlinear/nonlocal dispersion of the Camassa-Holm (CH) equation. This equation is one order more accurate in asymptotic approximation beyond KdV, yet it still preserves complete integrability via the inverse scattering transform method. Its traveling wave solutions contain both the KdV solitons and the CH peakons as limiting cases.

DOI: 10.1103/PhysRevLett.87.194501 PACS numbers: 47.35.+i, 05.45.Yv, 11.10.Ef, 11.10.Lm

Water wave theory first introduced solitons as solutions of unidirectional nonlinear wave equations, obtained via asymptotic expansions around simple wave motion of the Euler equations for shallow water in a particular Galilean frame [1]. Later developments identified some of these water wave equations as completely integrable Hamiltonian systems solvable by the inverse scattering transform (IST) method, see, e.g., [2]. We shall discuss the following 1 + 1 quadratically nonlinear equation in this class for unidirectional water waves with fluid velocity $u(x,t)$:

$$m_t + c_0 u_x + u m_x + 2 m u_x = -\gamma u_{xxx} . \qquad (1)$$

Here $m = u - \alpha^2 u_{xx}$ is a momentum variable, partial derivatives are denoted by subscripts, the constants α^2 and γ/c_0 are squares of length scales, and $c_0 = \sqrt{gh}$ is the linear wave speed for undisturbed water at rest at spatial infinity, where u and m are taken to vanish. (Any constant value $u = u_0$ is also a solution.) Equation (1) was first derived by using asymptotic expansions directly in the Hamiltonian for Euler's equations in the shallow water regime and was thereby shown to be bi-Hamiltonian and IST integrable in [3]. Its periodic solutions were treated in [4,5], and references therein. Before [3], families of integrable equations similar to (1) were known to be derivable from the theory of hereditary symmetries [6]. However, (1) was not written explicitly nor derived physically as a water wave equation, and its solution properties were not studied before [3]. See [7] for an insightful discussion of how the integrable equation (1) relates to the theory of hereditary symmetries.

The interplay between the local and nonlocal linear dispersion in this equation is evident in its phase velocity relation, $\omega/k = (c_0 - \gamma k^2)/(1 + \alpha^2 k^2)$, for waves with frequency ω and wave number k linearized around $u = 0$. For $\gamma/c_0 < 0$, short waves and long waves travel in the same direction. Long waves travel faster than short ones (as required in shallow water) provided $\gamma/c_0 > -\alpha^2$. Then the phase velocity lies in $\omega/k \in (-\gamma/\alpha^2, c_0]$.

Equation (1) is not Galilean invariant. Upon shifting the velocity variable by u_0 and moving into a Galilean frame $\xi = x - ct$ with velocity c, so that $u(x,t) = \tilde{u}(\xi,t) + c + u_0$, this equation transforms to

$$\tilde{m}_t + \tilde{u}\tilde{m}_\xi + 2\tilde{m}\tilde{u}_\xi = -\tilde{c}_0 \tilde{u}_\xi - \tilde{\gamma}\tilde{u}_{\xi\xi\xi} , \qquad (2)$$

with $\tilde{c}_0 = (c_0 + 2c + 3u_0)$, $\tilde{\gamma} = (\gamma - u_0\alpha^2)$, and appropriately altered boundary conditions at spatial infinity. Hence, we must regard Eq. (1) as a *family* of equations whose linear dispersion parameters c_0 and γ depend on the appropriate choice of Galilean frame and boundary conditions. The parameters c_0 and γ may even be *removed* by making such a choice, as for γ in [3]. In the following we will use only transformations that leave the boundary condition $u = 0$ at spatial infinity invariant.

This paper reports two main results. First, we identify how the dispersion coefficients for the linearized water waves appear as *parameters* in the isospectral problem for this IST-integrable equation. We also determine how the linear dispersion parameters α, c_0, and γ in (1) affect the isospectral content of its soliton solutions and the shape of its traveling waves. Second, we rederive Eq. (1) as a water wave equation and prove that it is correct to one order higher than for Korteweg-deVries (KdV) by using standard methods of asymptotic expansions and near-identity transformations. This new derivation and the analysis we present here attaches additional physical meaning to Eq. (1) in the context of asymptotics for shallow water wave equations. By means of a near-identity transformation it is shown that (1) is asymptotically equivalent to the fifth order KdV equation.

Equation (1) restricts to two separately integrable soliton equations for water waves. When $\alpha^2 \to 0$ this equation becomes the KdV equation

 0031-9007/01/87(19)/194501(4)$15.00

$$u_t + c_0 u_x + 3uu_x = -\gamma u_{xxx}, \tag{3}$$

which for $c_0 = 0$ has the famous soliton solution $u(x,t) = u_0 \mathrm{sech}^2[(x - ct)\sqrt{u_0/\gamma}/2]$, $c = c_0 + u_0$, see, e.g., [2]. Instead, choosing the Galilean frame for which $\gamma \to 0$ in Eq. (1) implies the Camassa-Holm (CH) equation

$$u_t + c_0 u_x - \alpha^2 u_{xxt} + 3uu_x = \alpha^2(2u_x u_{xx} + uu_{xxx}),$$

which for $c_0 = 0$ has the "peakon" soliton solutions $u(x,t) = ce^{-|x-ct|}$ discovered and analyzed in [3,8].

Equation (1) is bi-Hamiltonian and, hence, isospectral.—The term bi-Hamiltonian means the equation may be written in two compatible Hamiltonian forms, namely, as $m_t = -B_2(\delta H_1/\delta m) = -B_1(\delta H_2/\delta m)$ with

$$H_1 = \int u^2 + \alpha^2 u_x^2 \, dx,$$

$$B_2 = \partial_x m + m\partial_x + c_0\partial_x + \gamma\partial_x^3,$$

$$H_2 = \int u^3 + \alpha^2 uu_x^2 + c_0 u^2 - \gamma u_x^2 \, dx,$$

$$B_1 = \partial_x - \alpha^2\partial_x^3.$$

These bi-Hamiltonian forms restrict properly to those for KdV when $\alpha^2 \to 0$ and to those for CH when $\gamma \to 0$. Compatibility of B_1 and B_2 is assured since $\partial_x m + m\partial_x$, ∂_x, and ∂_x^3 are all mutually compatible Hamiltonian operators, see, e.g., [12]. From this viewpoint, γ and α^2 are deformation parameters for the *Riemann equation* $u_t + 3uu_x = 0$. No further deformations of these Hamiltonian operators involving higher order partial derivatives would be compatible with B_2 [12]. Related results are discussed in [13,14].

By the standard Gelfand-Dorfman theory [15], its bi-Hamiltonian property implies that the nonlinear equation (1) arises as a compatibility condition for two linear equations, namely, the *isospectral eigenvalue problem,*

$$\lambda\left(\frac{1}{4} - \alpha^2\partial_x^2\right)\psi = \left(\frac{c_0}{4} + \frac{m(x,t)}{2} + \gamma\partial_x^2\right)\psi, \tag{4}$$

and the *evolution equation* for the eigenfunction ψ,

$$\psi_t = -(u + \lambda)\psi_x + \frac{1}{2}u_x\psi. \tag{5}$$

Compatibility of these linear equations ($\psi_{xxt} = \psi_{txx}$) and isospectrality ($d\lambda/dt = 0$) imply Eq. (1). Consequently, the nonlinear water wave equation (1) admits the IST method for the solution of its initial value problem, just as the KdV and CH equations do. In fact, the isospectral problem for Eq. (1) restricts to the isospectral problem for KdV (i.e., the Schrödinger equation) when $\alpha^2 \to 0$, and it restricts to the isospectral problem for CH discovered in [3] when $\gamma \to 0$.

Defining a new spectral parameter $\mu^{-2} = \gamma + \lambda\alpha^2$ yields a spectral problem in the same form as for CH,

$$\left[\partial_x^2 - \frac{1}{4\alpha} + \mu^2\left(\frac{m}{2} + \frac{\gamma + c_0\alpha^2}{4\alpha^2}\right)\right]\psi = 0,$$

which is analyzed and discussed in [13,17]. The continuous spectrum lies in $\mu^{-2} \in [0, \gamma + c_0\alpha^2)$, which is non-negative for shallow water dispersion. In this form of the isospectral equation, the limit to KdV is singular.

Spectral content and solution behavior.—Provided m decreases sufficiently rapidly at spatial infinity, Eq. (4) has both continuous and discrete spectra. These spectral components correspond to the two different types of solution behavior available for Eq. (1). The continuous spectrum of the isospectral eigenvalue problem (4) spans the band of allowed linearized phase speeds, namely, $\lambda \in (-\gamma/\alpha^2, c_0)$. This continuous spectrum corresponds to radiation (linear waves). The discrete spectrum of (4) lies above this band, with $\lambda > c_0 \geq 0$. The discrete spectrum corresponds to the soliton sector of the solution space. This is also what is seen in numerical computations [18]. In the zero-dispersion limit for both $c_0 \to 0$ and $\gamma \to 0$, the corresponding isospectral problem for the CH equation has purely discrete spectrum representing only peakon solutions [3].

The derivation of (1) from shallow water wave asymptotics is similar to Whitham's derivation of the KdV equation [1], except that we keep terms of second order in the small parameters $\epsilon_1 = a/h$ and $\epsilon_2 = h^2/l^2$. Here $\epsilon_1 \geq \epsilon_2 > \epsilon_1^2$, a, h, and l denote the wave amplitude, the mean water depth, and a typical horizontal length scale (e.g., a wavelength), respectively. We shall sketch the derivation here and give details elsewhere [18]. We start with Laplace's equation for the velocity potential of an inviscid, incompressible, and irrotational fluid moving in a vertical plane under gravity with an upper free surface, as, e.g., in [1], p. 464. Length is measured in terms of l, height in h, and time in l/c_0. The elevation η is scaled with a, and fluid velocity u is scaled with $c_0 a/h$.

The result of the expansion is the equation for the surface elevation η [1], p. 466 (the higher order terms can, e.g., be found in [19]),

$$0 = \eta_t + \eta_x + \frac{3}{2}\epsilon_1\eta\eta_x + \frac{1}{6}\epsilon_2\eta_{xxx} - \frac{3}{8}\epsilon_1^2\eta^2\eta_x + \epsilon_1\epsilon_2\left(\frac{23}{24}\eta_x\eta_{xx} + \frac{5}{12}\eta\eta_{xxx}\right) + \epsilon_2^2\frac{19}{360}\eta_{xxxxx}. \tag{6}$$

Next, following Kodama [20,21] we shall apply the near-identity transformation, $\eta = u + \epsilon_1 f(u) + \epsilon_2 g(u)$, to the η equation (6) and seek functionals $f(u)$ and $g(u)$ such that the transformed equation is the integrable u equation (1), at order $O(\epsilon_2^2)$. Thus, we seek conditions for the existence of a near-identity transformation, such that $f(u)$ and $g(u)$ ensure the equivalence of Eqs. (6) and (1) at order $O(\epsilon_2^2)$. At this order, the functionals $f(u)$ and $g(u)$ should generate the terms uu_x, $u_x u_{xx}$, uu_{xxx}, and u_{xxx}. The allowable forms turn out to be u_{xx} for $g(u)$ and both u^2 and $u_x\partial^{-1}u$ for $f(u)$, where ∂^{-1} means integration in x. Hence, we set

$$\eta = u + \epsilon_1(\alpha_1 u^2 + \alpha_2 u_x\partial_x^{-1}u) + \epsilon_2\beta u_{xx}, \tag{7}$$

where α_1, α_2, and β are constants to be determined. Kodama's near-identity transformation thus implies the normal form of the u equation to which the η equation (6) is equivalent at $O(\epsilon_2^2)$. Other recent applications of this normal form approach are discussed in [21,22]. The physical interpretation [23] of the nonlocal term in the transformation is that it is responsible for a change in the phase shift of the two soliton solution of the original and transformed equations.

The transformation to (1) needs to produce a term of the form u_{xxt}. After applying the Kodama transformation and eliminating all higher time derivatives this term is obtained by applying the Helmholtz operator $I - \epsilon_2 \nu \partial^2$ to the resulting equation. Alternatively one could keep the u_{xxt} term that appears from the transformation and only partially transform its t derivative. The two procedures are equivalent. Application of the Helmholtz operator introduces another free parameter ν into the problem, which is necessary in order to obtain (1) in the end. In particular, it is only the special value $\nu = 19/60$ which removes the highest derivative of fifth order. In the remainder of the paper, ν is fixed to this value. With this choice, and with $\alpha_1 = 7/20$, $\alpha_2 = -1/5$, and $\beta = 1/30$, we obtain the equation

$$m_t + u_x + \frac{\epsilon_1}{2}(um_x + 2mu_x) + \epsilon_2 \frac{3}{20} u_{xxx} = 0, \quad (8)$$

where $m = u - \epsilon_2 \nu u_{xx}$.

In order to compare predictions, the solutions u must be transformed back to η using (7). However, since the derivation used not only the transformation (7) but also involved application of the Helmholtz operator, it is not clear that this is sufficient. The inverse transformation $u = u(\eta)$ of the same form as (7) with u and η interchanged is substituted into (8) in order to recover (6). We find that the coefficients just reverse their signs. We conclude that (1) is equivalent to the shallow water wave equation (6) up to, and including, terms of $\mathcal{O}(\epsilon_2^2)$.

Note that the removal of the highest order term was made possible by introducing the additional parameter ν in the Helmholtz operator. In [22] this term was removed by another term of the form xu_t in the Kodama transformation. This term, however, is not uniformly bounded and moreover changes the dispersion relation; we therefore discard its use. By using only the uniformly bounded terms in the Kodama transformation the shallow water equation (6) can be transformed into the integrable fifth order KdV equation [19].

Equation (8) is transformed to the fifth order KdV equation by the above transformation with the choice $\alpha_1 = 19/10$, $\alpha_2 = 19/20$, and $\beta = 19/30$. We conclude that (1) is asymptotically equivalent to the integrable fifth order KdV equation, and both of them are equivalent to (6), up to and including order $\mathcal{O}(\epsilon_2^2)$. The equivalence of (1) to the fifth order KdV equation breaks down in the peakon limit, $\gamma \to 0$, $c_0 \to 0$, because the transformation as well as the resulting equation contains terms divided by $\gamma + c_0\alpha^2$.

To recover (1) from (8) we reintroduce dimensional variables to find $\alpha^2 = h^2\nu$ and $\gamma = -3c_0h^2/20$. Hence, the normalized phase speed is

$$\frac{\omega}{kc_0} = \left(1 - \frac{1}{6}\frac{k^2h^2}{1 + \nu k^2h^2}\right). \quad (9)$$

The normalized phase speed for water waves is $\sqrt{\tanh(kh)/kh}$, see, e.g., [1]. Expanding this and the previous result for the phase speed in a Taylor series in wave steepness kh around $kh = 0$ yields agreement up to quintic order, provided $\nu = 19/60$, the previously found remarkable value of ν. The form of Eq. (1) originally appearing as Eq. (3) in [3] is recovered by setting $\nu = 1/3$, which is nearly the optimal value.

The traveling wave solution is obtained by the ansatz $u(x,t) = u(s)$, with $s = x - ct$, after which Eq. (1) can be integrated twice. The solution whose velocity vanishes at spatial infinity is given by

$$(c + \gamma/\alpha^2 - u)\alpha^2(du/ds)^2 = (c - c_0 - u)u^2. \quad (10)$$

When the limits of the radiation band coincide at zero, the peakon traveling wave equation reemerges. Otherwise, the traveling wave solution can be expressed implicitly as a quadrature, or in parametric form via a Sundmann transform of the independent variable $(ds/d\tau)^2 = (c + \gamma/\alpha^2 - u)\alpha^2$, as

$$u(\tau) = \hat{u}\,\mathrm{sech}^2 A\tau, \qquad A = \sqrt{\hat{u}}/2,$$

$$s(\tau) = 2\alpha\sqrt{D/\hat{u}}\,\sinh^{-1}\left(\frac{\sinh A\tau}{\sqrt{1 - \hat{u}/D}}\right) - 2\alpha\tanh^{-1}\left(1 + \frac{D/\hat{u} - 1}{\tanh^2 A\tau}\right)^{-1/2},$$

where $\hat{u} = c - c_0$ and $D = c_0 + \gamma/\alpha^2$. The curvature at the maximum is $-\frac{1}{2}\hat{u}^2/(\gamma + c_0\alpha^2) = -3\hat{u}^2/(c_0h^2)$, which is independent of ν. Introduction of ν broadens the width of the solution as compared to the pure sech^2 solution of KdV. For $c_0 \to 0$ and $\gamma \to 0$ this smooth family approaches the peakon solution. However, this limit cannot be attained for physical water waves because it implies vanishing mean depth, $h \to 0$.

In order to compare with experimental findings the solution has to be transformed from u back to η. To accomplish this, we expand u in a series of sech$^2 bs$, which gives

$$u(s) = a\,\mathrm{sech}^2 bs + \frac{19}{20}\epsilon_1 a^2\,\mathrm{sech}^4 bs,$$

where $b^2 = 3a\epsilon_1/(4\epsilon_2)$ and $c = 1 + \epsilon_1 a/2 + \frac{19}{40}\epsilon_1^2 a^2$ in $s = x - ct$. Applying the Kodama transformation gives

$$\eta(s) = a\left(1 + \frac{1}{2}\epsilon_1 a\right)\mathrm{sech}^2 bs + \frac{3}{4}\epsilon_1 a^2\,\mathrm{sech}^4 bs.$$

This is the same solution for (6) as found in [19] by solving the fifth order KdV equation (KdV5). This again shows that KdV5 and (1) are asymptotically equivalent. Normalizing the height and comparing to the KdV traveling wave shows that this wave form is slightly broadened and travels faster, in agreement with experimental findings [24].

Discussion.—The water wave equation (1) combines the elements of linear dispersion in KdV with the nonlinear, nonlocal dispersion of CH. This combination is IST integrable because it retains the bi-Hamiltonian and isospectral properties of the KdV and CH equations. The dispersion parameters α, c_0, and γ knit the KdV and CH equations into a family of integrable equations and also appear together in the eigenvalue problem (4).

As recognized already in [3], the dispersion parameters c_0 and γ in (1) are linked to each other by combined velocity shifts and Galilean boosts. Although these transformations preserve the isospectral property of the CH equation, they alter its isospectral content in two ways. First, they add to the discrete CH peakon spectrum a band of continuous spectrum near the origin. This isospectral band corresponds to linear radiation and spans the allowed range of linear phase velocity. Second, either c_0 or γ nonvanishing breaks the reflection-reversal symmetry of the CH equation for $c_0 = 0$ that allows coexistence of its generalized-function peakon and antipeakon soliton solutions traveling in opposite directions. Thus, adding linear dispersion breaks this discrete symmetry and removes the antipeakon solutions, while it also rounds the peaks of the peakon solutions; thereby, regularizing them into ordinary solitons, plus linear radiation.

This Letter confirmed that (1) is a genuine shallow water equation by rederiving it as the next equation beyond KdV at order $O(\epsilon_2^2)$ in an asymptotic expansion of the Euler equations. This is achieved by using a near identity transformation (7) due to Kodama [20] and by applying the Helmholtz operator. Thus, the equation originally discovered in [3] via asymptotic expansion of the Euler fluid Hamiltonian is recovered in the family of equations derived here for the case $\nu = 1/3$. The optimal value $\nu = 19/60$ makes the equation (i) integrable, (ii) asymptotically equivalent up to order $\mathcal{O}(\epsilon_2^2)$ to the integrable fifth order KdV equation and to the shallow water wave equation (6), and (iii) most accurate in its linear dispersion relation. An improved traveling wave for shallow water waves is calculated as a result.

We are grateful to R. Camassa, O. Fringer, I. Gabitov, R. Grimshaw, Y. Kodama, B. McCloud, J. Montaldi, P.J. Olver, T. Ratiu, M. Roberts, P. Swart, and A. Zenchuk for their constructive comments and encouragement. This work began at the MASEES 2000 Summer School in Peyresq, France. D. D. H. was supported by DOE, under contract No. W-7405-ENG-36. G. A. G. was supported by a European Commission Grant, contract No. HPRN-CT-2000-00113, for the Research Training Network *Mechanics and Symmetry in Europe* (MASIE). H. R. D. was also partially supported by MASIE.

*Email address: h.r.dullin@lboro.ac.uk
†Email address: g.gottwald@eim.surrey.ac.uk
‡Email address: dholm@lanl.gov

[1] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley Interscience, New York, 1974).
[2] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
[3] R. Camassa and D. D. Holm, Phys. Rev. Lett. **71**, 1661 (1993).
[4] M. Alber, R. Camassa, D. D. Holm, and J. E. Marsden, Lett. Math. Phys. **32**, 137 (1994).
[5] M. Alber, R. Camassa, V. N. Fedorov, D. D. Holm, and J. E. Marsden, Phys. Lett. A **264**, 171 (1999); (to be published).
[6] A. S. Fokas and B. Fuchssteiner, Nonlinear Anal. Theory Methods Appl. **3**, 849 (1981).
[7] B. Fuchssteiner, Physica (Amsterdam) **95D**, 229 (1996).
[8] Equation (1) also arises as the Euler-Poincaré equation for an averaged Lagrangian [9]. Its solutions describe geodesic motion with respect to the H_1 metric of u on the Bott-Virasoro Lie group [10]. The KdV equation arises the same way for the L_2 metric, see [11].
[9] D. D. Holm, J. E. Marsden, and T. S. Ratiu, Phys. Rev. Lett. **80**, 4173 (1998); D. D. Holm, J. E. Marsden, and T. S. Ratiu, Adv. Math. **137**, 1 (1998).
[10] G. Misiolek, J. Geom. Phys. **24**, 203 (1998).
[11] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry* (Springer, New York, 1999), 2nd ed.
[12] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York, 1993).
[13] A. S. Fokas, Physica (Amsterdam) **87D**, 145 (1995).
[14] A. S. Fokas, P. J. Olver, and P. Rosenau, in *Algebraic Aspects of Integrable Systems,* edited by A. S. Fokas and I. M. Gel'fand (Birkhäuser, Boston, 1996), pp. 93–101.
[15] As in [3], the isospectral problem for Eq. (1) may be obtained via the squared-eigenfunction approach from its bi-Hamiltonian property by using the methods of [16].
[16] I. M. Gelfand and I. Ya. R. Dorfman, Funct. Anal. Appl. **13**, 248 (1979).
[17] R. Beals, D. H. Sattinger, and J. Smigialsky, Adv. Math. **140**, 190 (1998).
[18] H. R. Dullin, G. Gottwald, and D. D. Holm (to be published).
[19] Li Zhi and N. R. Sibgatullin, J. Appl. Math. Mech. **61**, 177 (1997).
[20] Y. Kodama, Phys. Lett. **107A**, 245 (1985); **112A**, 193 (1985); **123A**, 276 (1987).
[21] Y. Kodama and A. V. Mikhailov, in *Algebraic Aspects of Integrable Systems: In Memory of Irene Dorfman,* edited by A. S. Fokas and I. Gelfand (Birkhäuser, Boston, 1996), pp. 173–204.
[22] A. S. Fokas and Q. M. Liu, Phys. Rev. Lett. **77**, 2347 (1996).
[23] Y. Kodama (private communication).
[24] P. D. Weidman and T. Maxworthy, J. Fluid Mech. **85**, 417 (1978).