Alpha Cluster Condensation in ^{12}C and ^{16}O

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(Received 29 June 2001; published 17 October 2001)

A new α-cluster wave function is proposed which is of the α-particle condensate type. Applications to ^{12}C and ^{16}O show that states of low density close to the 3 and 4 α-particle thresholds in both nuclei are possibly of this kind. It is conjectured that all self-conjugate $4n$ nuclei may show similar features.

DOI: 10.1103/PhysRevLett.87.192501 PACS numbers: 21.60.Gx, 03.75.Fi, 21.10.Gv, 27.20.+n

There exists an intriguing problem when bosonic clusters as bound states of fermions are produced, and the Bose character of the composite clusters competes with the fermionic properties of their constituents. As an example, we discuss the relevance of α-like four-nucleon correlations in atomic nuclei. Special attention is paid to such correlations which correspond to an α-type condensate in low-density symmetric nuclear matter, similar to the Bose-Einstein condensation observed for finite numbers of bosonic atoms such as Rb or Na in traps.

It is a well known fact that in light nuclei many states are of the cluster type [1–4]. In the case of cluster states of stable nuclei where we have only very few excess nucleons in addition to the clusters, they are all located close to or above the threshold energy of breakup into constituent clusters. This fact which is known as the threshold rule [5] means that the intercluster binding is weak in cluster states. The threshold rule can be considered as a necessary condition for the formation of the cluster structure, because if the intercluster binding is strong the clusters overlap strongly and the clusters will loose their identities.

For example, one of the fundamental questions of the cluster model is what kind of α-particle cluster states can be expected to exist around the threshold energy $E_{n\alpha}^{\rm thr} = nE_\alpha$ of $n\alpha$ breakup in self-conjugate $4n$ nuclei. One possible answer to this question, which is strongly under debate, is the existence of the cluster state of a linear $n\alpha$ chain structure. The idea of the linear α chain state, originally due to Morinaga [6], is so fascinating that recently the formation of linear 6α chain states in ^{24}Mg was studied extensively by experiments and also theoretically [7]. The possibility of the linear 3α chain state in ^{12}C, which is the simplest linear α chain state, was studied in detail by many authors solving the 3α problem microscopically [4]. However, these three-body studies all showed that the 3α-cluster states around the 3α threshold energy $E_{3\alpha}^{\rm thr}$ do not have a linear chain structure. For example, it was found that the calculated second 0^+ state in ^{12}C, which corresponds to the observed second 0^+ state located at 0.39 MeV above the 3α threshold energy, has a structure where α clusters interact predominantly in relative S waves. Thus it was concluded that the cluster state near $E_{3\alpha}^{\rm thr}$ has not a linear chain structure but rather an α-particle gaslike structure.

On the other hand, there have been recent theoretical investigations on the possibility of α-particle condensation in low-density nuclear matter [8,9]. Röpke *et al.* [8] made a variational ansatz for the solution of the in-medium four-body equation. Beyer *et al.* [9] solved the Faddeev-Yakubovsky equations for an α-like cluster in nuclear matter. The outcome of these studies was that such α condensation can occur only in the low-density region below a fifth of the saturation value. At higher densities rather a state of ordinary p-n, n-n, or p-p Cooper pairing will prevail. In view of these results it may be a tempting idea that in finite self-conjugate $4n$ nuclei one could expect the existence of excited states of dilute density composed of a weakly interacting gas of α particles. Since the α cluster is a Bose particle, such states could approximately be considered as an $n\alpha$ cluster condensed state and eventually excitations thereof.

The purpose of this Letter is to report on our study which not only confirms that indeed the second 0^+ state in ^{12}C could be considered as such a condensed state but that in addition in ^{16}O also such a state close to the threshold possibly exists. We then conjecture that the existence of such α-condensed states might be a general feature in $N = Z$ nuclei.

For the purpose of our study we write down a new type of α-cluster wave function describing an α-particle Bose condensed state:

$$|\Phi_{n\alpha}\rangle = (C_\alpha^\dagger)^n|\text{vac}\rangle, \tag{1}$$

where the α-particle creation operator is given by

$$C_\alpha^\dagger = \int d^3R\, e^{-\mathbf{R}^2/R_0^2} \int d^3r_1 \cdots d^3r_4 \times \varphi_{0s}(\mathbf{r}_1 - \mathbf{R})a_{\sigma_1\tau_1}^\dagger(\mathbf{r}_1) \cdots \varphi_{0s}(\mathbf{r}_4 - \mathbf{R})a_{\sigma_4\tau_4}^\dagger(\mathbf{r}_4), \tag{2}$$

where $\varphi_{0s}(\mathbf{r}) = (1/(\pi b^2))^{3/4}e^{-\mathbf{r}^2/(2b^2)}$ and $a_{\sigma\tau}^\dagger(\mathbf{r})$ is the creation operator of a nucleon with spin-isospin $\sigma\tau$ at the spatial point $\mathbf{r}$. The total $n\alpha$ wave function therefore can be written as

$$\langle \mathbf{r}_1\sigma_1\tau_1,\ldots,\mathbf{r}_{4n}\sigma_{4n}\tau_{4n}|\Phi_{n\alpha}\rangle \propto \mathcal{A}\{e^{-\frac{2}{B^2}(\mathbf{X}_1^2+\cdots+\mathbf{X}_n^2)}\phi(\alpha_1)\cdots\phi(\alpha_n)\}, \qquad (3)$$

where $B = (b^2 + 2R_0^2)^{1/2}$ and $\mathbf{X}_i = (1/4)\sum_n \mathbf{r}_{in}$ is the center-of-mass coordinate of the ith α cluster α_i. The internal wave function of the α cluster α_i is $\phi(\alpha_i) \propto \exp[-(1/8b^2)\sum_{m>n}^4(\mathbf{r}_{im} - \mathbf{r}_{in})^2]$. The wave function of Eq. (3) is totally antisymmetrized by the operator $\mathcal{A}$. It is to be noted that the wave function of Eqs. (1) and (3) expresses the state where $n\alpha$ clusters occupy the same $0s$ harmonic oscillator orbit $\exp[-\frac{2}{B^2}\mathbf{X}^2]$, where B is an independent variational width parameter. For example, if B is of the size of the whole nucleus, while b remains more or less at the free α-particle value (a situation encountered below), then the wave function (3) describes an $n\alpha$ cluster condensed state in the macroscopic limit $n \to \infty$. For finite systems we know from the pairing case that such a wave function still can more or less reflect Bose condensation properties. Of course the total center-of-mass motion can and must be separated out of the wave function of Eq. (1) for finite systems. In the limiting case of $B = b$ (i.e., $R_0 = 0$), Eq. (3) describes a Slater determinant of harmonic oscillator wave functions. We also point out that for $B \neq b$ the wave function (1),(3) is different from Brink's α-cluster state [2].

The state $|\Phi_{n\alpha}\rangle$ has spin parity 0^+. In the limit of $R_0 = 0$, the normalized wave function $|\Phi_{n\alpha}^N\rangle = |\Phi_{n\alpha}\rangle/\sqrt{\langle\Phi_{n\alpha}|\Phi_{n\alpha}\rangle}$ is identical to a harmonic oscillator shell model wave function with the oscillator parameter b. For $n = 3$ it is identical to the p-shell wave function $|(0s)^4(0p)^8, [444]0^+\rangle$, and for $n = 4$ it is identical to the double closed shell wave function, $|(0s)^4(0p)^{12}, 0^+\rangle$. This is easily proved by noticing that these limit wave functions for $n = 3$ and 4 have maximum spatial symmetry [444] and [4444], respectively. Only ^{8}Be has an α-particle structure in its ground state. Heavier $n\alpha$ nuclei collapse to the dense state in their ground state, but the individual α's may reappear when these nuclei are dilated, i.e., excited.

We calculated the energy surfaces in the two parameter space, R_0 and b, $E_{n\alpha}(R_0, b) = \langle\Phi_{n\alpha}^N(R_0, b) \times |\hat{H}|\Phi_{n\alpha}^N(R_0, b)\rangle$, for $n = 3$ and 4. The Hamiltonian $\hat{H}$ consists of the kinetic energy, the Coulomb energy, and the effective nuclear force named F1 which was proposed by one of the authors and contains a finite range three-nucleon force in addition to the finite range two-nucleon force [10]. This force reproduces reasonably well the binding energy and radius of the α particle, the α-α phase shifts of various partial waves, and the binding energy and density of nuclear matter. As we see below this force also gives good results for binding energies and radii of ^{12}C and ^{16}O.

In Figs. 1 and 2 we give the contour maps of the energy surfaces $E_{n\alpha}(R_0, b)$ for ^{12}C and ^{16}O. The qualitative features of both surfaces are similar. They show a valley running from the outer region with large $R_0 > 11$ fm and $b \approx b_\alpha = 1.44$ fm to the inner region with small R_0 and $b > b_\alpha$, where b_α is the oscillator parameter of the free α particle. The valleys have a saddle point at $R_0 \approx 10$ fm for $n = 3$ and at $R_0 \approx 10.6$ fm for $n = 4$. Beyond the saddle point, $E_{n\alpha}(R_0, b_\alpha) \approx E_{n\alpha}^{\rm thr} = nE_\alpha$, where $E_\alpha = -27.5$ MeV is the theoretical binding energy of the free α particle by the present F1 force in Hartree-Fock approximation. Therefore, we have $3E_\alpha = -82.5$ MeV and $4E_\alpha = -110$ MeV. The height of the saddle point measured from the theoretical threshold energy is about 1.4 MeV for $n = 3$ and 2.2 MeV for $n = 4$. The appearance of the saddle point is due to the increase of the Coulomb energy and kinetic energy towards the inward direction which is not yet compensated by the gain in potential energy around the saddle point region. This saddle point will help to stabilize the possible α condensed state around $E_{n\alpha}^{\rm thr}$. The minimum of the energy surface is located at $R_0 \approx 2$ fm for $n = 3$ and at $R_0 \approx 1$ fm for $n = 4$. Since $R_0 = 0$ means the shell model limit, we thus see that the wave function even at the energy minimum point deviates from the shell model limit and shows rather strong α-particle correlations. The gain in energy from the shell model limit is 10.3 MeV for ^{12}C and 4.7 MeV for ^{16}O. Before comparing numbers with experiments we have to make a quantum mechanical calculation. This is achieved via a standard Hill-Wheeler ansatz taking R_0 and b as the Hill-Wheeler coordinates. However, in order to reduce the complexity of the calculation and because the valleys run essentially parallel to the R_0 axis at $b = b_\alpha$, we take $b = b_\alpha =$ const and only discretize the R_0 variable. We therefore have

$$|\Psi_{n\alpha,k}\rangle = \sum_j f_k((R_0)_j, b_\alpha)|\Phi_{n\alpha}^N((R_0)_j, b_\alpha)\rangle. \qquad (4)$$

The normalization of $f_k((R_0, b)_j)$ is so that the kth eigenfunction $|\Psi_{n\alpha,k}\rangle$ is normalized. The adopted mesh size of R_0 values is typically 0.5 fm. In order to see the character of the obtained wave function $|\Psi_{n\alpha,k}\rangle$, we introduce the overlap amplitude

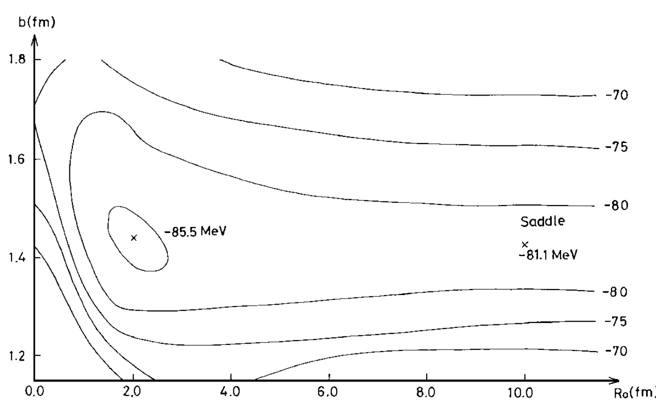

FIG. 1. Contour map of the energy surface $E_{3\alpha}(R_0, b)$ for ^{12}C. Numbers attached to the contour lines are the binding energies.

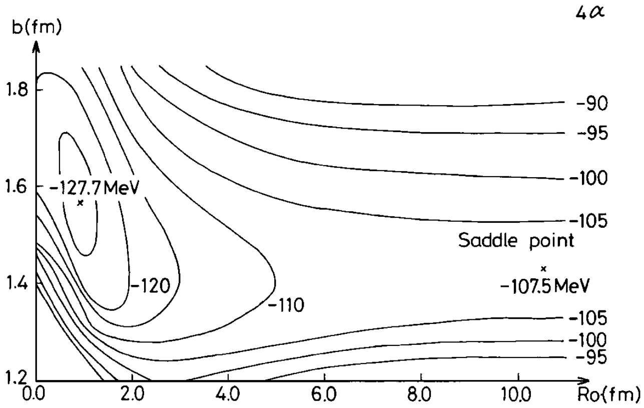

FIG. 2. Contour map of the energy surface $E_{4\alpha}(R_0, b)$ for ^{16}O. Numbers attached to the contour lines are the binding energies.

$$A_{n\alpha,k}(R_0, b) = \langle \Phi^N_{n\alpha}(R_0, b) | \Psi_{n\alpha,k} \rangle . \qquad (5)$$

From this overlap amplitude we can estimate the relevant values of the variational parameter R_0 in the different states k as is discussed below.

After outlining the results for the new kind of wave function for ^{12}C and ^{16}O, we discuss whether the obtained condensed states correspond to the states found in these nuclei. We first consider ^{12}C, i.e., $n = 3$; see Table I. The calculated lowest two eigenenergies are situated at -85.9 and -82.0 MeV. The lowest energy state corresponds to the ground state of ^{12}C and is only slightly lower than the minimum point of the energy surface located at -85.5 MeV. However, the calculated ground state is still above the observed binding energy of ^{12}C which is at -92.16 MeV. In order to reproduce the observed ^{12}C binding energy satisfactorily we have to extend our functional space so as to include the spatial symmetry broken wave functions. The second eigenvalue lies 0.36 MeV above our theoretical 3α threshold energy, $E^{\text{thr}}_{3\alpha} = -82.5$ MeV, and we believe that it corresponds to the observed second 0^+ state of ^{12}C which lies 0.5 MeV above $E^{\text{thr}}_{3\alpha}$. As seen in Table I the rms radius of the obtained wave function $|\Psi_{3\alpha,2}\rangle$ is 4.29 fm which is much larger than the one of the ground state which is 2.97 fm, slightly greater than the experimental value 2.45 fm but in agreement with the missing binding of 6.75 MeV. We thus see that the second 0^+ state corresponds to a very dilute system of average density which is only about a fifth of the experimental ground state density.

To characterize the wave function by a typical value of the width parameter R_0, we consider the overlap amplitude $A_{3\alpha,k}(R_0, b_\alpha)$ given in Eq. (5) as a function of R_0 at fixed b_α. Whereas the ground state ($k = 1$) wave function is almost exhausted by one $\Phi^N_{3\alpha}(R_0, b_\alpha)$ with $R_0 \approx 2$ fm, which is quite close to the wave function of the minimum energy point of the energy surface, the second $0^+_{k=2}$ state has the largest overlap amplitude (about 0.87) with $R_0 \approx 4.5$ fm. This rather large value implies that the distribution of the center-of-mass momenta is rather narrow, in a certain approximation to an α condensate in infinite nuclear matter where all α clusters populate the same state $P = 0$ of the center-of-mass momentum. The fact that the calculated $0^+_{k=2}$ state is of dilute density is in agreement with nuclear matter calculations [8,9] where it was shown that a condensate of α-like particles (quartetting) is possible only in matter with density $\rho \leq 0.03$ fm^{-3}. The average distance of the α clusters in the dilute $0^+_{k=2}$ state is in agreement with this value for low-density nuclear matter, where the overlap of the α clusters is small so that the Pauli blocking effects are weak.

Let us now discuss the case of ^{16}O, i.e., $n = 4$. The energies of the lowest observed 0^+ states are shown in Table II, together with the corresponding widths. The first excited 0^+_2 state at 6.06 MeV is very well known to have α-clustering character [1,4] and is well described by the ^{12}C + α microscopic cluster model as having the structure where the α cluster moves in an S state around the ^{12}C cluster in its ground state [12] though other cluster states also have been proposed [3]. Similarly, the third excited 0^+_4 state at 12.05 MeV can be described by the same model where the α cluster moves in a D state around the ^{12}C cluster in its first 2^+ excited state [12]. We exclude these well understood states from our further discussion. The excited states 0^+_3 at 11.26 MeV and 0^+_5 at 14.0 MeV observed in ^{12}C + α elastic scattering [11] cannot be described by such a model. Furthermore, they have very large decay widths, not typical for the other states. These

TABLE I. Comparison of the generator coordinate method calculations with experimental values. $E^{\text{thr}}_{n\alpha} = nE_\alpha$ denotes the threshold energy for the decay into α clusters; the values marked by * correspond to a refined mesh (see main text).

		E_k (MeV)	E_{exp} (MeV)	$E_k - E^{\text{thr}}_{n\alpha}$ (MeV)	$(E - E^{\text{thr}}_{n\alpha})_{\text{exp}}$ (MeV)	$\sqrt{\langle r^2 \rangle}$ (fm)	$\sqrt{\langle r^2 \rangle_{\text{exp}}}$ (fm)
^{12}C	$k = 1$	-85.9	-92.16 (0^+_1)	-3.4	-7.27	2.97	2.65
	$k = 2$	-82.0	-84.51 (0^+_2)	$+0.5$	0.38	4.29	
	$E^{\text{thr}}_{3\alpha}$	-82.5	-84.89				
^{16}O	$k = 1$	-124.8 (-128.0)*	-127.62 (0^+_1)	-14.8 (-18.0)*	-14.44	2.59	2.73
	$k = 2$	-116.0	-116.36 (0^+_3)	-6.0	-3.18	3.16	
	$k = 3$	-110.7	-113.62 (0^+_5)	-0.7	-0.44	3.97	
	$E^{\text{thr}}_{4\alpha}$	-110.0	-113.18				

TABLE II. Observed excitation energies $E_{\rm exc}$ and widths Γ of the lowest five 0^+ excited states in ^{16}O.

	$E_{\rm exc}$ (MeV)	Γ (MeV)
0_2^+	6.06	
0_3^+	11.26	2.6
0_4^+	12.05	1.6×10^{-3}
0_5^+	14.0	4.8
0_6^+	14.03	2.0×10^{-1}

states may be described by our new wave function as condensed states.

As seen from Table I, the experimental value of the ground state (0_1^+) of ^{16}O at -127.62 MeV is well reproduced by the calculated energy value for the ground state ($0_{k=1}^+$) at -124.8 MeV. The calculated energy is above the minimum energy of the energy surface. It is because the b value of the minimum energy point is fairly larger than b_α and the minimum energy point is not covered by the adopted mesh points. In order to have a better wave function for the ground state we, of course, need to include mesh points around the minimum energy in our generator coordinate calculation. When we adopt $b = 1.57$ fm which is the b value of the energy minimum point, the generator coordinate calculation gives -128.0 MeV as the lowest eigenenergy. The rms radius of the calculated ($0_{k=1}^+$) state is 2.59 fm and is slightly smaller than the observed (0_1^+) rms radius, 2.73 fm, of ^{16}O. The second ($0_{k=2}^+$) state of our calculation is bound by 6 MeV below the theoretical 4α threshold energy. The rms radius of this state is 3.12 fm and this state has the largest overlap amplitude (about 0.86) with $\Phi_{4\alpha}^N(R_0, b_\alpha)$ with $R_0 \approx 2.5$ fm. We conjecture that this state corresponds to the observed 0_3^+ state situated at 3.18 MeV below the observed 4 α threshold energy. Indeed one may argue that there will be some mixing between the second ($0_{k=2}^+$) state and the ^{12}C + α state, bringing theoretical and experimental energies closer together. The third ($0_{k=3}^+$) state of our calculation is bound by 0.7 MeV below the theoretical 4α threshold energy. We think that it may correspond to the measured 0_5^+ state situated at 0.44 MeV below the observed 4α threshold energy. This state has a very large rms radius of 3.94 fm and has the largest overlap amplitude (about 0.80) with $\Phi_{4\alpha}^N(R_0, b_\alpha)$ with $R_0 \approx 4.1$ fm. In analogy to the case of ^{12}C, these values indicate that this state of dilute density should be considered as just the 4α-cluster condensed state that we expected. One should point out that the ease with which we get the 0^+ states around $E_{n\alpha}^{\rm thr}$ is a strong indication that our wave function (1),(3) grasps the essential physics, because otherwise the threshold states are very difficult to obtain. We also mention that the present formalism yields very good results for the ground state of ^{8}Be as well.

In conclusion, our present study thus predicts in ^{12}C and ^{16}O the existence of near-$n\alpha$-threshold states which are the finite system analogs to α-cluster condensation in infinite matter. They are characterized by low-density states so that the α clusters are not strongly overlapping and by an n-fold occupation of their identical S-wave center-of-mass wave function. Therefore, these states are quite similar in structure to the Bose-Einstein condensed states of bosonic atoms in magnetic traps where all atoms populate the same lowest S-wave quantum orbital. Because of the short lifetime of the α-condensed states, the predicted large values for the rms radii may be verified by indirect methods. The measurements of the spectra of the emitted α particles should allow us to determine the Coulomb barrier which is expected to be small for the low-density states. Of particular interest would be the α-α coincidence measurement of decaying condensed states.

We conjecture that such condensed α-cluster states near the $n\alpha$ threshold may also occur in other heavier $4n$ self-conjugate nuclei. For example, condensed 6α states of ^{24}Mg could be deformed and the measurement of a reduced moment of inertia over the rigid body value would be a strong indication for α-particle superfluidity. The wave function we have proposed in this work is very flexible and can straightfowardly be adopted for the description of other condensation phenomena, such as ordinary Cooper pairing or a mixture of Cooper pair and α-particle condensation.

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