

## Dynamic Instability of a Rotating Bose-Einstein Condensate

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We consider a Bose-Einstein condensate subject to a rotating harmonic potential, in connection with recent experiments leading to the formation of vortices. We use the classical hydrodynamic approximation to the nonlinear Schrödinger equation to determine almost analytically the evolution of the condensate. We predict that this evolution can exhibit dynamical instabilities, for the stirring procedure previously demonstrated at ENS and for a new stirring procedure that we put forward. These instabilities take place within the range of stirring frequency and amplitude for which vortices are produced experimentally. They provide therefore an initiating mechanism for vortex nucleation.

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Quantized vortices in superfluid helium II, in particular the issue of vortex nucleation in a rotating container, have long been the subject of intense work [1]. With the recent production of gaseous Bose-Einstein condensates [2], the subject has gained a renewed interest. On the experimental side, three groups have succeeded in obtaining vortices in atomic condensates, with two different techniques: a phase imprinting technique at JILA [3] and the equivalent of the helium rotating bucket experiment at ENS [4] and at MIT [5]. At ENS, a rotating laser beam superposed onto the magnetic trap holding the atoms creates a harmonic rotating potential with adjustable anisotropy  $\epsilon$  and rotation frequency  $\Omega$ . For a well chosen range of variation for  $\Omega$  one or several vortices are nucleated, and then detected as holes in the density profile of the gas after ballistic expansion [4] or by a measurement of the angular momentum of the condensate [6]. A striking feature of the ENS experimental results is that, for a very weak anisotropy  $\epsilon$ , nucleation of vortices takes place in a narrow interval of rotation frequencies  $[\Omega_{\min}, \Omega_{\max}]$  around  $0.7\omega_{\perp}$ , where  $\omega_{\perp}$  is the mean oscillation frequency of the atoms in the  $x$ - $y$  plane, whatever the number of atoms or the oscillation frequency  $\omega_z$  along  $z$  in the experiment [7].

While the JILA experiment is well understood theoretically [8], the situation is more involved for the ENS experiment. Several theoretical articles, inspired by the case of superfluid helium, have tried to predict the value of the lower vortex nucleation frequency  $\Omega_{\min}$  from purely thermodynamic arguments [9–15]. The proposed values for  $\Omega_{\min}$  are significantly different from the observed value of  $0.7\omega_{\perp}$ , or depend on the trap aspect ratio  $\omega_z/\omega_{\perp}$  or on the atom number, in contradiction with the observations at ENS. Also, thermodynamical reasonings are not able to predict the upper vortex nucleation frequency  $\Omega_{\max}$ , which is also close to  $0.7\omega_{\perp}$  for low anisotropy  $\epsilon$ .

In this Letter, we consider the time dependent problem of a condensate subject to a harmonic stirring potential. We use the classical hydrodynamic approximation to the time dependent Gross-Pitaevskii equation (GPE), an approximation well justified for the ENS parameters [16]. We are then able to reformulate the partial differential hydrody-

amic equations in terms of ordinary differential equations, which allows an almost analytical solution [17]. Our main result is the discovery of dynamical instabilities in the evolution of the condensate for a certain range of the rotation frequency and of the trap anisotropy. These instabilities will invalidate the classical hydrodynamic approximation after some evolution time. We have checked with a numerical solution of the Gross-Pitaevskii equation that vortices then enter the condensate.

The existence of such a dynamical instability explains why in earlier numerical work the time dependent Gross-Pitaevskii equation was found to nucleate vortices [13,18,19]. Furthermore, the instability range that we predict is very close to the experimentally observed range of vortex nucleation, for various stirring procedures. For the stirring procedure of [4,6], we recover the “universal” numerical value 0.7 for  $\Omega/\omega_{\perp}$  leading to vortex nucleation for low anisotropies  $\epsilon$ . We provide a simple physical interpretation of this value: For  $\Omega/\omega_{\perp} = 1/\sqrt{2} \approx 0.7$  the harmonic stirring potential resonantly excites a quadrupole mode of the condensate [20], which induces large oscillations of the condensate and eventually a dynamical instability sets in. We also investigate a new excitation procedure to nucleate vortices, which has recently been implemented at ENS [21]: The rotation of the stirring potential is set up very slowly. The gas then follows adiabatically a branch of steady state until the branch becomes dynamically unstable. The corresponding lower rotation frequency  $\Omega$  that we predict for vortex nucleation is also very close to the experimental value.

In our model, atoms are trapped in a harmonic potential rotating at the instantaneous frequency  $\Omega(t)$  around the  $z$  axis. For convenience, all the calculations of this paper are done in a rotating frame where the trap axes are fixed. The trapping potential then reads

$$U(\vec{r}, t) = \frac{1}{2} m \omega_{\perp}^2 \left\{ [1 - \epsilon(t)] x^2 + [1 + \epsilon(t)] y^2 + \left( \frac{\omega_z}{\omega_{\perp}} \right)^2 z^2 \right\}, \quad (1)$$

where  $m$  is the mass of an atom, and  $\epsilon(t)$  is the trap anisotropy at time  $t$ . The parameters  $\omega_\perp$  and  $\omega_z$  are the oscillation frequencies of the atoms in transverse and axial directions for vanishing anisotropy of the stirring potential. Within the mean-field approximation, the time evolution of the condensate field or macroscopic wave function  $\psi(\vec{r}, t)$  can be described by the time dependent Gross-Pitaevskii equation [22]:

$$i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \vec{\nabla}^2 + U(\vec{r}, t) + g|\psi|^2 - \Omega(t)\hat{L}_z \right] \psi, \quad (2)$$

where  $g = 4\pi\hbar^2 a/m$  is the coupling constant, proportional to the  $s$ -wave scattering length  $a$  of the atoms, here taken to be positive, and where the inertial term proportional to the angular momentum operator  $\hat{L}_z$  along the  $z$  axis accounts for the frame rotation. The condensate field  $\psi$  can be written in terms of density  $\rho$  and phase  $S$ :

$$\psi(\vec{r}, t) = \sqrt{\rho(\vec{r}, t)} e^{iS(\vec{r}, t)/\hbar}. \quad (3)$$

The equation obtained from the GPE for  $\rho$  is just the continuity equation. The equation for  $S$  contains the so-called quantum pressure term  $\hbar^2 \vec{\nabla}^2 \sqrt{\rho}/2m\sqrt{\rho}$  that we neglect here as compared to the mean-field term  $\rho g$  in the Thomas-Fermi approximation. We obtain

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\text{div} \left[ \rho \left( \frac{\vec{\nabla} S}{m} - \vec{\Omega}(t) \times \vec{r} \right) \right], \quad (4) \\ -\frac{\partial S}{\partial t} &= \frac{(\vec{\nabla} S)^2}{2m} + U(\vec{r}, t) + g\rho - [\vec{\Omega}(t) \times \vec{r}] \cdot \vec{\nabla} S. \quad (5) \end{aligned}$$

A very fortunate feature of the harmonic trap is that these superfluid hydrodynamic equations can be solved exactly for a condensate initially at equilibrium in the nonrotating trap with the following quadratic ansatz for the condensate density and phase [23]:

$$\rho_c(\vec{r}, t) = \rho_0(t) + \frac{m\omega_\perp^2}{g} \sum_{i,j=1}^3 x_i A_{ij}(t) x_j, \quad (6)$$

$$S_c(\vec{r}, t) = s_0(t) + m\omega_\perp \sum_{i,j=1}^3 x_i B_{ij}(t) x_j, \quad (7)$$

where  $x_1$ ,  $x_2$ , and  $x_3$  are the coordinates along the  $x$ ,  $y$ , and  $z$  axes, respectively. The time dependent dimensionless coefficients  $A_{ij}$  and  $B_{ij}$  form  $3 \times 3$  symmetric matrices  $A$  and  $B$  which from Eqs. (4) and (5) obey the evolution equations:

$$\omega_\perp^{-1} \frac{dA}{dt} = -2A \text{Tr}B - 2\{A, B\} + \frac{\Omega}{\omega_\perp} [R, A], \quad (8)$$

$$\omega_\perp^{-1} \frac{dB}{dt} = -2B^2 - W - A + \frac{\Omega}{\omega_\perp} [R, B], \quad (9)$$

where  $\{, \}$  stands for the anticommutator,  $[, ]$  stands for the commutator of two matrices, the matrix  $W$  is diago-

nal, with components  $W_{11} = [1 - \epsilon(t)]/2$ ,  $W_{22} = [1 + \epsilon(t)]/2$ , and  $W_{33} = (\omega_z/\omega_\perp)^2/2$ , and the matrix  $R$ , originating from the vectorial product in  $\hat{L}_z$ , has vanishing elements except for  $R_{12} = -R_{21} = 1$  [24]. Note that these equations do not depend on the number of atoms nor on the coupling constant  $g$ .

In a first stage, it is very important to study steady state solutions of the above equations. We restrict ourselves to solutions that have the same symmetry as the initial state: even parity along  $z$ , this parity being preserved by time evolution. We then find a unique class of solutions, reproducing the results of [25]: The condensate phase varies as  $S(\vec{r}) = m\omega_\perp \beta xy$ , where  $\beta$  is a real root of

$$\beta^3 + \left(1 - 2\frac{\Omega^2}{\omega_\perp^2}\right)\beta - \frac{\Omega}{\omega_\perp} \epsilon = 0. \quad (10)$$

The steady state matrix  $A$  is diagonal with elements given in [25]. We have plotted in Fig. 1 the values of  $\beta$  as functions of the rotation frequency  $\Omega$  for a fixed anisotropy  $\epsilon$ . For  $\Omega$  between zero and a bifurcation value  $\Omega_{\text{bif}}(\epsilon)$  depending on  $\epsilon$ , there is a single branch of solution for  $\beta$ . This branch is supplemented by two extra branches when  $\Omega > \Omega_{\text{bif}}(\epsilon)$ .

We now turn back to the time dependent problem. Clearly, a condensate with a vortex cannot be described within the quadratic ansatz [(6) and (7)] as the phase  $S_c$  corresponds to an irrotational velocity flow. The actual scenario for the vortex nucleation that we put forward is the following: Initially very small deviations  $\delta\rho(\vec{r}, t)$  of the condensate density and  $\delta S(\vec{r}, t)$  of the condensate phase from the quadratic shapes  $\rho_c$  and  $S_c$  may grow exponentially fast in the course of time evolution, eventually leading the condensate to a structure very different from Eqs. (6) and (7). This may happen when a dynamical instability is present.

To reveal such an instability, we obtain, from the evolution Eqs. (4) and (5), linearized equations of motion for

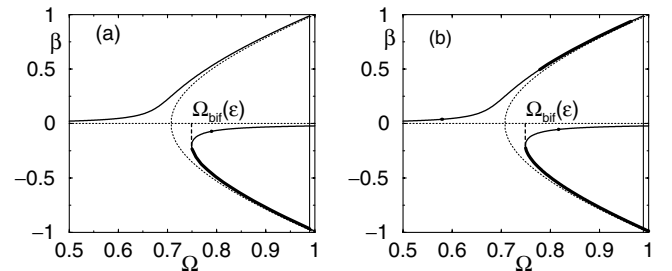


FIG. 1. Phase parameter  $\beta$  for a steady state condensate as a function of the rotation frequency  $\Omega$  in units of  $\omega_\perp$ . Dotted lines:  $\epsilon = 0$ . Solid lines:  $\epsilon = 0.022$ . For  $\epsilon = 0.022$  and  $\omega_z/\omega_\perp = 0.1$ , the thick lines on the curves indicate where the solution has a dynamical instability (a) of degree  $n = 2$  and (b) of degree  $n = 3$ . The vertical line is the border of the center of mass instability domain for  $\epsilon = 0.022$ .

initially small deviations  $\delta\rho$  and  $\delta S$  from  $\rho_c$  and  $S_c$ :

$$\frac{D\delta\rho}{Dt} = -\text{div}\left(\rho_c \frac{\vec{\nabla}\delta S}{m}\right) - \delta\rho \frac{\vec{\nabla}^2 S_c}{m}, \quad (11)$$

$$\frac{D\delta S}{Dt} = -g\delta\rho. \quad (12)$$

In these equations, we have introduced the convective derivative  $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v}_c(\vec{r}, t) \cdot \vec{\nabla}$ , where  $\vec{v}_c = \vec{\nabla} S_c/m - \vec{\Omega} \times \vec{r}$  is the velocity field of the condensate in the rotating frame. A polynomial ansatz for  $\delta S$  and  $\delta\rho$  of an arbitrary total degree  $n$  in the coordinates  $x$ ,  $y$ , and  $z$  solves these linear equations exactly [26]. This is another nice consequence of the harmonicity of the trap. In practice, we calculate the evolution operator  $\mathcal{U}_n(t)$  mapping the coefficients of the polynomials at time zero onto their values after a time evolution  $t$ . Dynamical instability takes place when one or several eigenvalues of  $\mathcal{U}_n$  grow exponentially fast with time  $t$ . Note that, after rescaling of the variables, Eqs. (11) and (12) become independent of the number of atoms and of the coupling constant  $g$ , in a way similar to Eqs. (8) and (9).

Now we perform a linear stability analysis for two different stirring procedures of the condensate.

*Procedure I.*—The ellipticity of the stirring potential  $\epsilon$  is kept fixed and the rotation frequency  $\Omega(t)$  of the stirrer is very slowly ramped up from zero to its final value. The condensate, initially in steady state with a vanishing parameter  $\beta$ , adiabatically follows the upper branch of steady states with  $\beta \geq 0$  (see Fig. 1). It is then sufficient to determine the dynamic stability of the upper branch of steady states. This greatly simplifies the calculation as one just has to identify eigenmodes of Eqs. (11) and (12) evolving in time as  $\exp(\lambda t)$ , where the eigenvalue  $\lambda$  is a complex number. Dynamical instability takes place when  $\lambda$  can have a strictly positive real part. As shown in Fig. 1 for  $\epsilon = 0.022$ , we find that the upper branch for  $\beta$  is stable for modes of degree  $n = 2$  but presents two instability intervals for the modes of degree  $n = 3$ , a very narrow interval around  $\Omega = 0.58\omega_\perp$  and a broader interval starting at  $\Omega = 0.778\omega_\perp$ .

We have investigated in a systematic way the instability range of the upper branch of steady states, by varying the anisotropy  $\epsilon$ , the rotation frequency  $\Omega$ , and the degree  $n$ . The instability domain in the  $\Omega - \epsilon$  plane for modes of degree 3 is mainly made of a crescent, and the inclusion of higher degree modes ( $n = 4, 5$ ) add extra crescents from above (see Fig. 2a). Each crescent has on the  $\epsilon = 0$  axis (i) a broad basis at  $\Omega > \omega_\perp/\sqrt{2}$ , with a nonzero instability exponent, and (ii) a very narrow edge at  $\Omega = \omega_\perp/\sqrt{n} < \omega_\perp/\sqrt{2}$ , with a vanishing instability exponent. We show in Fig. 2b that the maximal instability exponent for  $n = 3$  has a remarkably weak dependence on  $\omega_z/\omega_\perp$ .

*Procedure II.*—This is the original experimental scenario of [4,6], where the stirring potential is rotated at a fixed frequency and the ellipticity of the stirrer is turned

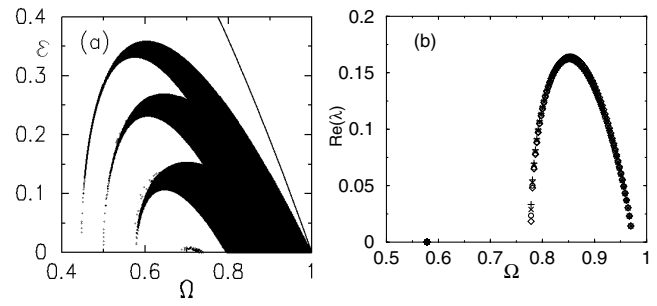


FIG. 2. For the upper branch of steady state condensates: (a) Dark areas: instability domain in the  $\Omega - \epsilon$  plane for  $\omega_z/\omega_\perp = 0.1$  for degrees  $n$  equal to 3, 4, and 5 (crescents from bottom to top). There is no dynamical instability for  $n \leq 2$ . Solid line: border  $\Omega^2 = (1 - \epsilon)\omega_\perp^2$  of the branch existence domain. (b) Maximal instability exponent  $\text{Re}(\lambda)$  for  $n = 3$  as a function of  $\Omega$ , for  $\epsilon = 0.022$ , and  $\omega_z/\omega_\perp = 0.1$  ( $\diamond$ ), 0.5 ( $\circ$ ), 1.0 ( $\times$ ), and  $\sqrt{8}$  ( $+$ ).  $\Omega$  and  $\text{Re}(\lambda)$  are in units of  $\omega_\perp$ .

on from zero to its final value  $\epsilon_f$  abruptly. In this case, we cannot rely on adiabatic following for the condensate density and phase, so we solve the time dependent Eqs. (8) and (9) for  $\rho_c(t)$  and  $S_c(t)$ . Then we perform a linear stability analysis as discussed above: We evolve a generic polynomial ansatz of degree  $n$  for the fluctuations  $\delta\rho$  and  $\delta S$  according to Eqs. (11) and (12), which allows one to construct the evolution operator  $\mathcal{U}_n(t)$  and to calculate  $Z_{\text{max}}(t)$ , the eigenvalue of  $\mathcal{U}_n(t)$  with the largest modulus. Then we define the mean instability exponent  $\text{Re}\langle\lambda\rangle$  as the mean slope of  $\ln|Z_{\text{max}}(t)|$  as a function of time.

This reveals that within a certain range of rotation frequency the system becomes dynamically unstable (see the solid line in Fig. 3). In the limit of a low anisotropy  $\epsilon$ , the

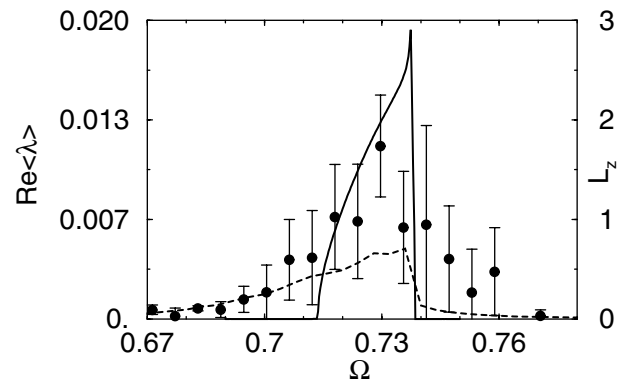


FIG. 3. For the stirring procedure II, with  $\epsilon_f = 0.01$  and  $\omega_z/\omega_\perp = 0.1$ : Solid line: mean instability exponent  $\text{Re}\langle\lambda\rangle$  (see text) of the vortex-free classical hydrodynamic solution  $\rho_c, S_c$  as a function of  $\Omega$ , for  $n = 3$ . Dashed line: mean angular momentum per particle  $\bar{L}_z$  (see text) obtained from  $\rho_c, S_c$ . Filled disks: experimentally measured angular momentum  $L_z$  per particle in the condensate after vortices have possibly entered the condensate [7]. The initial steady state condensate in the calculation of  $\bar{L}_z$  has a chemical potential  $\mu = 10\hbar\omega_\perp$ , close to the experimental value.  $\text{Re}\langle\lambda\rangle$  and  $\Omega$  are in units of  $\omega_\perp$ , and  $L_z, \bar{L}_z$  are in units of  $\hbar$ .

instability sets in when the rotation frequency  $\Omega$  is close to the value  $\approx 0.7\omega_{\perp}$ : In the lab frame, the stirring potential of frequency  $2\Omega$  is then resonant with a quadrupole mode [16,27] of the condensate of frequency  $\sqrt{2}\omega_{\perp}$ , and induces large amplitude oscillations of the condensate, resulting in a dynamical instability. More precisely, the condensate described by the quadratic ansatz  $\rho_c, S_c$  has an angular momentum oscillating in time around a nonzero value  $\bar{L}_z$ . This value  $\bar{L}_z$  is a peaked function of the rotation frequency  $\Omega$ , shown in the dashed line in Fig. 3. The peak of  $\bar{L}_z$  is not exactly located at  $\Omega = 0.7\omega_{\perp}$  because of nonlinear effects in Eqs. (8) and (9). The peak structure of the instability exponent in Fig. 3 is similar to the peak structure of  $\bar{L}_z$ , with a narrower width as dynamical instability of the vortex free solution  $\rho_c, S_c$  sets in for the higher values of  $\bar{L}_z$  only. For values of  $\Omega$  significantly above or below  $0.7\omega_{\perp}$ , the stirrer is out of resonance with the quadrupole mode and induces only small and stable oscillations of the condensate. For larger values of  $\epsilon$ , the instability interval in  $\Omega$  broadens. We have also checked that the instability interval depends weakly on  $\omega_z/\omega_{\perp}$ .

What is the connection between the dynamical instabilities found here and the nucleation of vortices? To obtain a theoretical answer to this question, one has to go beyond a linear stability analysis to determine the evolution of the condensate in the long run: For a few values of the rotation frequency  $\Omega$  and for procedures I and II, we have checked by a numerical integration of the time dependent GPE in three dimensions, that vortices are indeed nucleated in the predicted instability domains: after some evolution time, the angular momentum in the numerical solution suddenly becomes larger than the classical hydrodynamic prediction, as vortices enter the condensate. An experimental answer to this question for the stirring procedure I has been provided recently at ENS [21]: The first clear evidence of a vortex appears for a rotation frequency  $\Omega = 0.77\omega_{\perp}$ , very close to our prediction  $\Omega = 0.778\omega_{\perp}$ . The agreement is also very good for procedure II as shown in Fig. 3: Our instability domain in  $\Omega$  coincides with the experimental vortex nucleation interval within a few percent.

In summary, the dynamical instabilities that we have identified provide an initiating mechanism for the production of vortices in a condensate stirred by a harmonic potential, in excellent agreement with the experimental results at ENS.

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