Theoretical Interpretation of Alfvén Cascades in Tokamaks with Nonmonotonic q Profiles

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Alfvén spectra in a reversed-shear tokamak plasma with a population of energetic ions exhibit a quasiperiodic pattern of primarily upward frequency sweeping (Alfvén cascade). Presented here is an explanation for such asymmetric sweeping behavior which involves finding a new energetic particle mode localized around the point of zero magnetic shear.

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The presence of energetic particles in a plasma can alter its behavior from that predicted by conventional magnetohydrodynamics (MHD) theory in two ways. First these particles can perturbatively destabilize a basic MHD mode. Alternatively, a sufficient number of these particles can nonperturbatively alter the very structure of the MHD modes. This latter behavior is relevant to certain shear Alfvénic perturbations often called energetic particle modes (EPM) [1–3]. In addition, in recent years there has been a great deal of interest in plasmas with reversed magnetic shear profiles, where transport and MHD stability properties have been shown to improve [4,5]. It is important for fusion experiments in shear reversed fields to understand the collective properties associated with energetic particles. Experiments in JT-60U [6] and JET [7] have investigated reversed shear regimes and have produced energetic particles with ion cyclotron heating (ICRH) [8]. Alfvén modes emerge in these experiments but their spectrum is often puzzling. This paper presents an example of how a purely MHD description is incompatible with the data while a description which accounts for the nonperturbative energetic particle response explains a large part of the data. The interpretation suggests a sensitive method to experimentally determine $q_{\min}$ (the minimum safety factor) in reversed magnetic shear tokamaks.

The JET experiments exhibit upward frequency sweeping phenomena, named Alfvén wave cascades (ACs) [9] (see Fig. 1a). Each cascade consists of several modes with different toroidal mode numbers and different frequencies. The toroidal mode numbers vary from $n = 1$ to $n = 6$. The frequency starts from 20–40 kHz and increases up to 100–120 kHz which is the toroidal Alfvén eigenmode (TAE) gap frequency. Similar data were obtained some time ago on JT-60U [6]. In both the JET and JT-60U data, the modes with higher toroidal mode numbers exhibit a more rapid frequency sweeping, and the higher n modes re-occur more often than the lower n modes. It is striking that downward frequency sweeping either does not appear, or appears only rarely. In both JET and JT-60U experiments, the minimum value of q decreases in time and a population of energetic ions is created by ICRH heating.

ACs resemble the global Alfvén eigenmode [10,11], whose frequency is close to the local value of the Alfvén wave frequency at the zero shear point in minor radius, $r = r_0$, i.e., $2\pi f_{\mathrm{AC}} \approx \omega_A(r_0) \equiv |k_\parallel(r_0)|V_A(r_0)$, where V_A is Alfvén velocity and $k_\parallel$ is the wave-vector component along the equilibrium magnetic field B_0. To avoid strong damping, the frequency f_{AC} needs to be somewhat larger than $\omega_A(r_0)$ if $\omega_A(r)$ has a maximum at $r = r_0$ and smaller than $\omega_A(r_0)$ if $\omega_A(r)$ has a minimum there. Otherwise, continuum resonance inhibits mode excitation by a moderate population of energetic ions.

In the standard theory of the global Alfvén eigenmode, the mode is associated with a minimum of the local Alfvén frequency $\omega_A(r)$ [10,11]. However, from both analytic considerations, and numerical calculations with the use of ideal MHD CSCAS code [12] we infer that a maximum of $\omega_A(r)$ is needed to explain the data. This conclusion follows from the local shear Alfvén wave dispersion relation which is $\omega_A/V_A = |k_\parallel(r)| = |n - m/q(r)|/R$, where m is the poloidal mode number. We impose the convention that ω_A and n are positive, and positive m is required to allow the mode frequency to be smaller than the TAE frequency, $f_{\mathrm{TAE}} = V_A/(4\pi qR)$. It is readily established that $\omega_A(r_0)$ is a maximum at $q = q_{\min}$ when $k_\parallel(r_0) < 0$ and a minimum when $k_\parallel(r_0) > 0$. We now assume that $q_{\min}$ decreases in time as it does in the JET and JT-60U experiments.

If the modes in the experiment just trace the Alfvén dispersion relation at $q = q_{\min}$, and $q_{\min}$ decreases, the frequency would increase in time when $k_\parallel(r_0) < 0$ and decrease in time when $k_\parallel(r_0) > 0$. This pattern is shown in Fig. 1b obtained from the CSCAS code that is applied to a series of JET experimental equilibria. In these CSCAS runs only the toroidal n number is a precise quantum number, while the dominant poloidal mode number m changes in steps as $q_{\min}$ decreases in time. This code automatically transfers the dominance of the mth poloidal harmonic to $m - 1$ as $q_{\min}$ passes through $q_{\mathrm{TAE}} = (m - 1/2)/n$ to

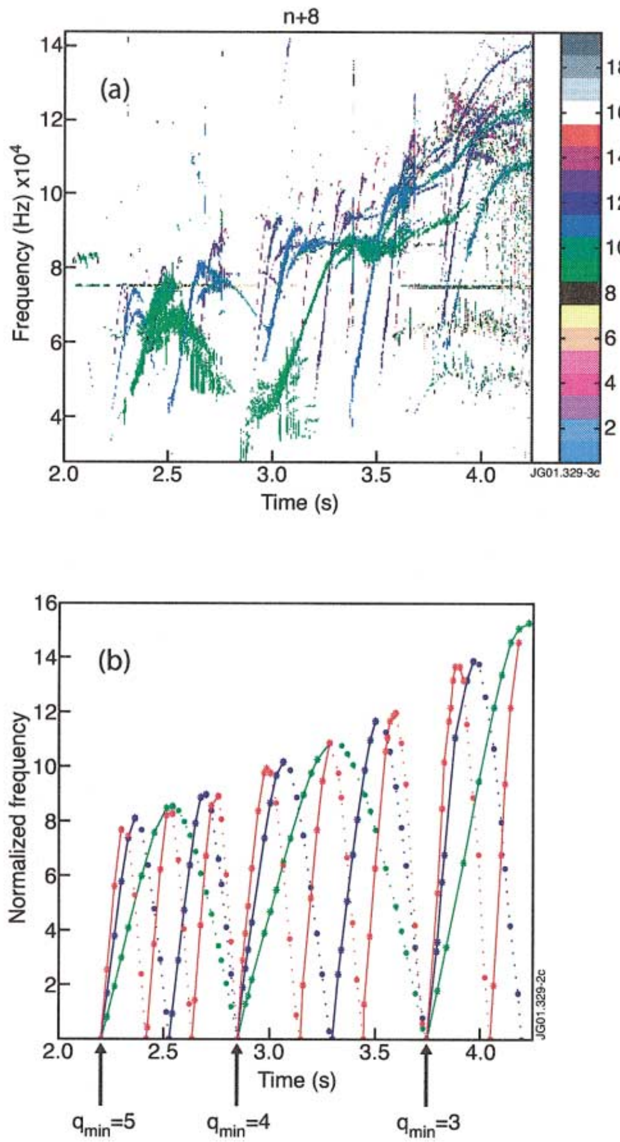

FIG. 1 (color). (a) Spectrogram of the magnetic perturbations, measured by the external Mirnov coils in JET plasma with nonmonotonic $q(r)$ (pulse No. 49382). Alfvén cascades of toroidal mode numbers from $n = 1$ through $n = 6$ are observed at frequencies below TAE frequency range, $f_{ac} \cong 30-100$ kHz $< f_{TAE}$. The vertical legend color codes the quantity $n + 8$. (b) The CSCAS analysis of temporal evolution of the normalized frequency $\omega_A(r_0)R/V_A$ at $q = q_{min}$ as $q_{min}(t)$ varies. Mode numbers plotted are $n = 1$ (green), $n = 2$ (blue), and $n = 3$ (red). Solid curves indicate local maxima of the Alfvén continuum, broken curves indicate local minima of the Alfvén continuum.

keep the mode frequency below the gap associated with the TAE frequency. It is also clear that the Alfvén continuum modes in Fig. 1b form bunches when q_{min} takes on integer values such as 3, 4, and 5. In the experiment the emerging modes also appear in bunches.

It is important to note that the transition from m to $m - 1$ changes the sign of $k_\parallel(r_0)$, which reverses the direction of frequency sweeping. However, the modes with downward sweeping are suppressed in the experiment. Thus a mechanism is needed that gives preference to the waves with negative $k_\parallel(r_0)$. The only way we have found to explain the asymmetry is to describe the fast particles response in a nonperturbative manner. This means that the fast particle contribution to the MHD equations affects the very existence of the mode rather than just the mode growth rate. We have also examined other candidate mechanisms. They all give preference to waves with positive $k_\parallel(r_0)$, which is inconsistent with the experiment.

Technically, the following feature of the deeply reversed shear discharges in JET is essential for our interpretation: the fast particle ∇B drift rate across the mode structure is faster than the bounce frequency or the mode frequency ω. As a result of fast drift, the hot particle response is found to be spatially local, which simplifies our analysis considerably. In this aspect, our theory is substantially different from past theories for the EPM which deal with the nonlocal hot particle response [1,2].

For a typical energy of fast ions ~500 keV, and with other plasma parameters chosen to be compatible with the relevant equilibrium, the fast particle orbits are found to be nonstandard. Indeed, our numerical calculations with the particle-following code HAGIS [13] and the CASTOR-K code [14] show that the toroidal drift frequency exceeds the poloidal bounce frequency and the orbit width is a substantial fraction of the plasma radius. These features make it easy to satisfy the condition that the ∇B drift frequency exceeds the eigenmode frequency for high n values, and marginally for $n = 1$.

Our formal derivation of the relevant energetic particle mode is based on the reduced MHD description of shear Alfvén perturbations and the drift kinetic description of energetic particles. We consider a low-beta plasma in a large-aspect-ratio torus, for which the perturbed vector potential $\delta \boldsymbol{A}$ and the perturbed fields $\delta \boldsymbol{E}$ and $\delta \boldsymbol{B}$ for the shear Alfvén wave can be represented by a single scalar function in the following forms: $\delta \boldsymbol{A} = \nabla \delta \Phi - \frac{\boldsymbol{B}}{B^2}(\boldsymbol{B} \cdot \nabla \delta \Phi)$, $\delta \boldsymbol{E} = -\frac{1}{c}\frac{\partial \delta \boldsymbol{A}}{\partial t}$, and $\delta \boldsymbol{B} = \nabla \times \delta \boldsymbol{A}$, where $\boldsymbol{B}$ is the equilibrium magnetic field. The equation for $\delta\Phi$ follows from a derivation procedure presented in Ref. [15], from which we find

$$\nabla \cdot \frac{1}{V_A^2 B^2}(\boldsymbol{B} \times [\nabla \delta \ddot{\Phi} \times \boldsymbol{B}]) = (\boldsymbol{B} \cdot \nabla)\frac{1}{B^2}\nabla \cdot \left[\boldsymbol{B} \times \left\{\nabla\left[\frac{1}{B^2}(\boldsymbol{B} \cdot \nabla \delta \Phi)\right] \times \boldsymbol{B}\right\}\right] - \left\{\nabla\left[\frac{1}{B^2}(\boldsymbol{B} \cdot \nabla \delta \Phi)\right] \cdot \Delta \boldsymbol{B}\right\} - \nabla \cdot \frac{4\pi}{B^2}[\boldsymbol{B} \times \delta \boldsymbol{F}], \quad (1)$$

where $\delta \boldsymbol{F}$ is the force density due to the perturbed anisotropic pressure that can be calculated with the use of the kinetic guiding center theory [16]. It was shown in Ref. [15] that the following relation holds for $\delta \boldsymbol{F}$ in Eq. (1) in the limit of low beta and large aspect ratio:

$$-\nabla \cdot \frac{1}{B^2}[\boldsymbol{B} \times \delta \boldsymbol{F}] = \frac{e}{c} \int d^3 v \, (\boldsymbol{v}_D \cdot \nabla \delta f) . \quad (2)$$

Here e is the energetic particle charge, $\boldsymbol{v}_D$ is the magnetic field gradient and curvature drift velocity, and the gradient ∇ operates on the perturbed distribution function δf with energy w and magnetic moment μ held fixed. Equation (2) generally involves summation over all species, but we will keep only the response from the ICRF heated energetic ions since one can show that the contribution from the background plasma pressure is relatively small in our ultimate eigenmode equation.

In order to evaluate the right-hand side of Eq. (2) we use the linearized drift kinetic equation neglecting equilibrium electric fields,

$$\frac{\partial \delta f}{\partial t} + v_\parallel \boldsymbol{b} \cdot \nabla \delta f + \boldsymbol{v}_D \cdot \nabla \delta f + [\delta(v_\parallel \boldsymbol{b})] \cdot \nabla f + (\delta \boldsymbol{v}_D) \cdot \nabla f + \boldsymbol{v}_{\boldsymbol{E}\times\boldsymbol{B}} \cdot \nabla f + \frac{v_\parallel}{\Omega}\left[\boldsymbol{b} \times \frac{\partial \delta \boldsymbol{b}}{\partial t}\right] \cdot \nabla f + \left[\mu \frac{\partial \delta B}{\partial t} + e(v_\parallel \boldsymbol{b} \cdot \delta \boldsymbol{E} + \boldsymbol{v}_D \cdot \delta \boldsymbol{E})\right] \frac{\partial f}{\partial w} = 0 , \quad (3)$$

with $w \equiv m v_\parallel^2/2 + \mu B$, $\boldsymbol{v}_D \equiv \boldsymbol{b} \times [\mu \nabla \boldsymbol{B} + m v_\parallel^2 (\boldsymbol{b} \cdot \nabla)\boldsymbol{b}]/m\Omega$, and $\boldsymbol{v}_{\boldsymbol{E}\times\boldsymbol{B}} \equiv \frac{c}{B^2}\, \delta \boldsymbol{E} \times \boldsymbol{B}$. Further Ω is the fast particle gyrofrequency, and the symbol δ denotes a perturbation of a quantity. Note that the integrand in Eq. (2) is exactly the third term on the left-hand side of Eq. (3). We will limit our consideration to the case of sufficiently fast drift velocity $\boldsymbol{v}_D$, as discussed in the introduction. Then the third term in Eq. (3) is the only term involving δf that need be retained. This simplification will lead to a differential equation rather than an integral equation for determining $\delta \Phi$. For the JET experiment under consideration this approximation is marginally good at $n = 1$ and improves for larger values of n. Further simplifications occur when we use $\boldsymbol{b} \cdot \delta \boldsymbol{B} = 0$ and $\boldsymbol{b} \cdot \delta \boldsymbol{E} = 0$ for shear Alfvén perturbations. We also take into account that $\frac{v_\parallel}{\Omega B}[\boldsymbol{b} \times \frac{\partial \delta \boldsymbol{B}}{\partial t}] \cdot \nabla f \ll \frac{v_\parallel}{B}\, \delta \boldsymbol{B} \cdot \nabla f$ since the mode frequency is much smaller than the gyrofrequency. In addition, we neglect $\delta \boldsymbol{v}_D \cdot \nabla f$. Then, with the elimination of some other small terms Eqs. (2) and (3) yield

$$\nabla \cdot \left(\frac{1}{B^2} \boldsymbol{B} \times \delta \boldsymbol{F}\right) = \frac{e}{B}\, \delta \boldsymbol{E} \times \boldsymbol{B} \cdot \nabla \frac{\int f \, d^3 v}{B} + \frac{e}{c}\, \delta \boldsymbol{B} \cdot \nabla \frac{\int v_\parallel f \, d^3 v}{B} - \frac{e}{B^2}[\delta \boldsymbol{E} \times \boldsymbol{B} \cdot (\boldsymbol{b} \cdot \nabla)\boldsymbol{b}] \int f \, d^3 v , \quad (4)$$

where we have transformed independent variables in the distribution function from $\boldsymbol{r}$, μ, and w to $\boldsymbol{r}$, $v_\parallel$, and $v_\perp$. Equations (1) and (4), together with $\delta \boldsymbol{E}$ and $\delta \boldsymbol{B}$ lead to a single equation of a form, $\hat{L} \delta \Phi = 0$, where $\hat{L}$ is a linear differential operator. In a torus, this operator is a periodic function of poloidal angle θ. Therefore, poloidal Fourier components of $\delta \Phi$ are generally coupled in the solution of this equation. However, in the case of nonmonotonic q profile in the presence of energetic particles the construction of essentially "cylindrical" modes is allowable if their frequencies are not too close to the TAE gap frequency. Formally, this means that we average all coefficients in $\hat{L}$ over θ. We then seek a cylindrical solution of the form $\delta \Phi = \psi(r) \exp(-i\omega t + in\varphi - im\theta)$, where φ is the toroidal angle and $\psi(r)$ is the radial eigenfunction. A straightforward averaging procedure with the added assumptions of $m \gg 1$ and large aspect ratio equilibrium with circular flux surfaces, gives the following equation for $\psi(r)$:

$$\frac{m^2}{r^2}\left(\frac{\omega^2}{V_A^2} - k_\parallel^2\right)\psi - \frac{\partial}{\partial r}\left(\frac{\omega^2}{V_A^2} - k_\parallel^2\right)\frac{\partial \psi}{\partial r} = -\frac{4\pi e}{cB} \frac{m}{r} \psi \frac{\partial}{\partial r}\left[\omega \langle n_h \rangle - k_\parallel \left\langle \frac{1}{e} j_{\parallel h} \right\rangle\right]. \quad (5)$$

where the subscript "h" denotes fast particles and the angular brackets denote flux surface averaging. The parallel wave number, $k_\parallel = \frac{1}{R}(n - \frac{m}{q(r)})$, can be expanded about the point $r = r_0$ where $q = q_{\min}$ (the point of zero shear). In the vicinity of r_0, we have

$$k_\parallel^2 = \frac{1}{R^2}\left(n - \frac{m}{q_{\min}}\right)^2 + \frac{1}{R^2} \frac{m q''_{\min}}{q_{\min}^2} (r - r_0)^2 \left(n - \frac{m}{q_{\min}}\right). \quad (6)$$

We assume that this expression for $k_\parallel$ is accurate over a region Δr where the mode is localized. Thus we require $(\Delta r)^2 < |n q_{\min} - m| \frac{q_{\min}}{m q''_{\min}}$. We can then replace ω and $k_\parallel$ on the right-hand side of Eq. (5) by the lowest order expressions, $\omega = \omega_A \equiv \frac{V_A}{R}|n - \frac{m}{q_{\min}}|$ and $k_\parallel = \frac{1}{R}(n - \frac{m}{q_{\min}})$, respectively. Then defining the dimensionless radial independent variable $x \equiv m(r - r_0)/r_0$ and a new dependent variable $\Psi(x) = \psi(x)(S + x^2)^{1/2}$, we obtain,

$$\Psi - \frac{\partial}{\partial x}\frac{\partial \Psi}{\partial x} = \Psi \frac{Q}{(S + x^2)} - \Psi \frac{S}{(S + x^2)^2} \quad (7)$$

with

$$S \equiv \frac{\omega^2 - \omega_A^2}{\omega_A^2} m \frac{q_{\min}}{q''_{\min} r_0^2} (m - n q_{\min});$$

$$Q \equiv -\frac{4\pi e R q_{\min}^2}{c B r_0 q''_{\min}} \frac{\partial}{\partial r} \times \left[\frac{V_A |m - n q_{\min}|}{(m - n q_{\min})} \langle n_h \rangle + \left\langle \frac{1}{e} j_{\parallel h} \right\rangle \right]_{r=r_0}.$$

It should be noted that the eigenvalue S has to be positive to avoid singularity in $\Psi(x)$ that would induce strong continuum damping. However, with Q neglected, a positive S does not give a radially localized eigenmode. To obtain one, a positive Q is required that exceeds a certain critical value Q_{cr}. Indeed, if Q is negative or zero, a "Schrödinger potential well" does not exist in Eq. (7).

An analysis of Eq. (7) establishes that $Q_{cr} = 1/4$ and that there is an infinite number of modes for $Q > Q_{cr}$. Approximate analytic solutions can be obtained for $Q - Q_{cr} \ll 1$ and $Q \gg 1$ and in between Eq. (7) has been solved numerically. The detailed analysis and results will be presented in a later publication. Here we note that if $Q - Q_{cr} \ll 1$, the value of S is given by $S = \exp[-2l\pi/(Q - 1/4)^{1/2}]$ where l is a positive integer, while if $Q \gg 1$, we find $S = Q - (2l + 1)Q^{1/2}$ (assuming the second term much less than the first term). For $Q = 1$, the numerically evaluated eigenvalue is $S = 0.1003$ for the longest wavelength mode. Note that S is relatively small even for $Q = 1$. The scale length of the longest wavelength mode is $\Delta r \approx S^{1/2} r_0/m$ for $S \ll 1$ and $\Delta r \approx Q^{1/4} r_0/m$ for $Q \gg 1$. We expect that dissipative processes will suppress the short wavelength modes with $Q - Q_{cr} \ll Q_{cr}$, which may require $Q \gtrsim 1$ to allow the energetic particle drive to excite these modes in the experiment.

We now make additional remarks about experimental implications of our calculations. We have concluded that in order for the frequency to sweep upward we need $m > n q_{\min}$ and that to have the mode we need $Q > 1/4$ (though in practice a larger Q value is required). There is strong bias in the expression for Q that favors $\partial \langle n_h \rangle / \partial r < 0$ near $r = r_0$ in order to meet the above two requirements. Then the frequencies of the allowed modes ($m > n q_{\min}$) increase as $q_{\min}$ decreases in time, whereas the condition for frequency decrease ($m < n q_{\min}$) is incompatible with mode existence. Further, we note that Q_{cr} is independent of m and n, a very satisfying result as many modes are characteristic for Alfvén cascades.

Once the existence of a mode is established, the mode growth rate, associated with resonant energetic particles, can be calculated with the use of straightforward perturbation theory. Also, the weakly nonlinear regime of mode saturation can be straightforwardly analyzed.

In summary the energetic particle mechanism described here is the only viable option we find to explain the observed Alfvén cascades. We conclude that the emerging bunches at times $t = 2.2$ s, 2.8 s, and 3.7 s arise when $q_{\min}$ is 5, 4, and 3, respectively. One can see the $n = 1$ mode emerging at these times together with higher n modes. The $n = 2$ mode has an extra appearance between the $n = 1$ bursts, and the $n = 3$ mode has two additional appearances between the $n = 1$ bursts, etc. Our identification of $q_{\min}$ correlates with the time behavior of the upper cascade frequency, which is close to the TAE frequency $V_A/4\pi R q_{\min}$.

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