Perturbative Finiteness in Spin-Foam Quantum Gravity

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The Lorentzian "normalized balanced state sum model" of quantum general relativity is finite on any nondegenerate triangulation. It provides a candidate for a background independent, perturbatively finite, quantum theory of general relativity in four dimensions and with Lorentzian signature.

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Quantum gravity remains a major open problem. The need of a "background independent" theory is now widely recognized: in such a theory, spacetime itself is dynamical, reflecting the dynamical character of physical geometry discovered with general relativity (GR). But the means of realizing background independence are currently being debated. In particular, it is still unclear whether the infinities appearing in the standard perturbative expansion of quantum GR are to be imputed to the fact that quantum GR is inconsistent at high energies, such as Fermi 4-fermion theory (and thus must be corrected, say by supersymmetry, higher spacetime dimensions, strings, etc.); or rather to the fact that it is the perturbation expansion over a smooth geometry that fails near the Planck scale. The second possibility is reinforced by the kinematical results of nonperturbative "loop" quantum gravity [1], according to which the Planck scale structure of physical spacetime is very far from smoothness [2]. To address the issue directly, one needs a formulation in which transition amplitudes in quantum GR could be computed, perhaps order by order in some expansion, but without any assumed background spacetime. Here, we present a fully background independent formulation of a Lorentzian quantum theory of gravity in 4D, and a proof of finiteness for the terms appearing in its perturbative expansion. The theory was introduced in [3] on the basis of the "balanced state sum model" defined in [4]. These results could reinforce the hope that quantum general relativity (QGR) might exist in 4D.

The theory is constructed by utilizing the duality between field theory on a group manifold (group field theory, or GFT) and QGR. This duality was first observed in [5,6] and expresses the remarkable fact that the amplitude of individual Feynman graphs of certain GFT's turn out to be exactly equivalent (that is, given by the same sum of identical terms) to the balanced state sums of Refs. [4,7] defined on a triangulation related to the graph. In turn, the balanced state sum is, presumably, a cutoff formulation of QGR on a fixed triangulation. Therefore the full Feynman expansion of the GFT might provide a well-defined version of the Wheeler-Misner-Hawking [8] "sum over 4-geometries" formulation of QGR. The cutoff on the number of degrees of freedom introduced by the triangulation is removed by the Feynman sum.

The Feynman expansion of the n-point functions of the GFT turns out to be a sum over spacetimes *with boundaries,* with the 3-geometry fixed on the boundaries. Therefore the n-point functions of the GFT directly give the physical 3-geometry to 3-geometry transition amplitudes of QGR. These quantities are 4D diffeomorphism invariant and, in principle, capture the physical content of quantum gravity [9].

This mechanism is a 4D analog of the duality between matrix models and quantum gravity in 2D, utilized some time ago to provide a nonperturbative definition of string theory in a 0D target space [10]. In both cases, the sum over geometries is generated as a sum of Feynman diagrams of an auxiliary theory: the matrix model in 2D, the GFT in 4D. In the 4D case, a Feynman graph of the GFT, colored with the labels determined by the kinematics of the theory, defines a "spin foam" [11–13], that is, a 2-complex labeled with representations of the internal gauge group. The spin foam has a natural interpretation as a (quantized) spacetime. As in the matrix models case, the use of an auxiliary field theory to generate spacetimes dynamically, realizes the full background independence of the spacetime theory. GFT's of the kind considered here were in fact first studied in 3D and 4D by Boulatov and Ooguri [14] as a (highly nontrivial) generalization of the matrix model idea. Unlike the Boulatov and Ooguri theories, which are dual to the *BF topological* theory, the GFT we use here is dual to a full fledged gravitational theory, with local degrees of freedom. The analogy with the matrix models is only partial, since short scale behavior and low energy limit are different in the two cases.

A balanced state sum was first introduced in [7] as a cutoff quantization of *Euclidean* QGR. The denomination "balanced" stems from the fact that only "balanced," or "simple" representations of the internal gauge group appear in the state sum. The model was made finite by passing from the representations of $so(4)$, to its q deformation at a root of unity, in analogy with the way Ponzano-Regge 3D quantum gravity is made finite in the Turaev-Viro

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model [15]. In [16], a better motivated normalization for the model was derived, and it was conjectured that with this new normalization the model (the "Euclidean normalized balanced state sum") was finite even without passing to the quantum group. The same normalization was considered, from a different perspective, in [17]. The finiteness conjecture was proven in [18]. In [4], a *Lorentzian* signature version of the state sum model was proposed, and it was again suggested that finiteness could be achieved passing to the quantum Lorentz algebra [19]. In [3], a change of normalization similar to the Euclidean case was proposed for the Lorentzian model. The resulting "Lorentzian normalized balanced state sum model" is in fact *finite* on any nondegenerate triangulation of a 4-manifold (in fact, on any nonsingular 2-complex).

Finiteness can be proven with mathematical rigor. The result is nontrivial: the individual terms of the sum are integrals over infinite domains of terms which are traces of operators on infinite dimensional balanced representations of the Lorentz algebra. The finiteness proof relies on technology developed in [20]. The finiteness of the individual terms comes about by representing them as multiple integrals on hyperbolic space and implementing a simple regularization. The finiteness of the complete state sum depends then on a delicate relationship between asymptotic estimates.

The balanced state sum is a term in the perturbation expansion of the GFT and there is only a finite number of such terms at every order of perturbation theory. Therefore, the finiteness of the balanced state sum implies that the GFT is perturbatively finite (up to the issue of singular 2-complexes, which appear in the expansion as well). Thus—up to this issue—the GFT introduced in [3] defines a background independent formulation of Lorentzian 4D quantum gravity, in which physical transition amplitudes can be computed by means of a perturbative expansion whose terms are finite at every order. The aim of this Letter is to present this theory to a large audience and to present the structure of the finiteness proof. Details of the proof will appear elsewhere. We discuss in the closure the relation of this construction with canonical loop quantum gravity, as well as the various aspects of this construction which are still unclear.

Consider a field $\phi(g_l)$, symmetric under permutation of its four argument, where $l = 1 \dots 4$ and $g_l \in G = \mathrm{SL}(2, C)$. Define the projectors P and R as

$$P\phi(g_l) \equiv \int_G dg\, \phi(g_l g), \qquad R\phi(g_l) \equiv \int_{U^4} du_l\, \phi(g_l u_l), \tag{1}$$

U is the SU(2) subgroup of $\mathrm{SL}(2, C)$ that leaves a fixed timelike vector invariant. dg and du are the corresponding invariant measures. The action of the GFT [3] is

$$S[\phi] = \int_{G^4} dg_l [P\phi(g_l)]^2 + \frac{\lambda}{5!} \int_{G^{10}} dg_{ij} \prod_i PR\phi(g_{ij}), \tag{2}$$

where $i, j = 1 \dots 5$, $i \neq j$, and $g_{ij} = g_{ji}$. Without the R projector (and with a compact group G), this action reduces to the Ooguri one [14], dual to BF theory.

The Feynman expansion in λ of the n-point functions of the theory was derived in [3]. It turns out to be given by a sum over 2-complexes σ with 5-valent vertices and 4-valent edges, of an amplitude $\mathcal{A}(\sigma)$. The reason Feynman graphs becomes 2-complexes in a GFT is explained in detail in [16]. The boundaries of the 2-complex correspond to the open legs of the Feynman graph.

As in the Peter-Weyl theorem, going to momentum space replaces group variables with indices in unitary representations. Denote the principal series unitary representations of the Lorentz algebra as $R(k, \rho)$, where k is an integer and ρ is a nonnegative real. The R projection kills all representations except the balanced ones. Balanced representations are those with $k = 0$ and are labeled by ρ. Label each face f of σ with a balanced representation of the Lorentz algebra, or simply with a positive parameter ρ_f. The explicit computation of the Feynman amplitude of the graph associated to σ yields

$$\mathcal{A}(\sigma) = \int_0^\infty d\rho_f \prod_f \rho_f^2 \prod_e \Theta(\rho_l^e) \prod_v I(\rho_{ij}^v). \tag{3}$$

The integration is over the labels of all the faces not belonging to the boundary. The products run over the faces f, the edges e, and the vertices v of σ, respectively. (ρ_l^e) label the four faces adjacent to the edge e; (ρ_{ij}^v) label the ten faces adjacent to the vertex v.

The functions Θ and I are traces of recombination diagrams for the balanced representations. These are called relativistic spin networks. The function Θ is given by the first diagram in Fig. 1. It was discovered to play a role in the model in [3] and its evaluation is in [4]. The function I is the interaction vertex, the essential element of the theory, and is given by the second diagram in Fig. 1.

As shown in [4], relativistic spin networks can be explicitly expressed as multiple integrals on the upper sheet H of the 2-sheeted hyperboloid in Minkowski space. To this purpose, define the projector kernel

$$K_\rho(x, y) = \sin[\rho d(x, y)]\{\rho \sinh[d(x, y)]\}^{-1}, \tag{4}$$

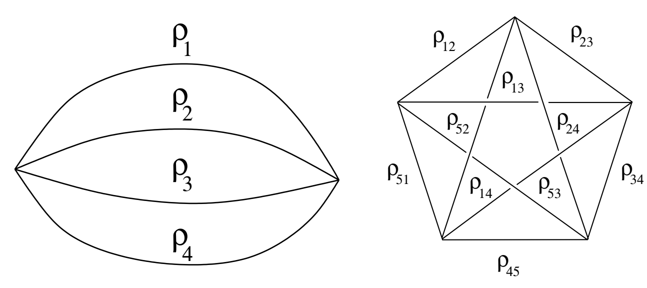

FIG. 1. The Θ and the I spin nets.

where $d(x,y)$ is the hyperbolic distance between x and y. The trace of a recombination diagram is given by a multiple integral of products of K's: one integral over H per each node, of the product on one kernel per each link. The integral is regularized by dropping one of the integrations. By Lorentz symmetry, the result is independent of the point not integrated over. Explicitly, Θ and I are given by

$$\Theta = \frac{1}{2\pi^2}\int_H K_{\rho_1}(x,y)\cdots K_{\rho_4}(x,y)\,dy\,, \tag{5}$$

$$I = \frac{1}{2\pi^2}\int_{H^4}\prod_{i\leq j=1,5} K_{\rho_{ij}}(x_i,x_j)\,dx_1\,dx_2\,dx_3\,dx_4\,. \tag{6}$$

Equations (3)–(6) define the Feynman amplitude completely.

This amplitude is precisely the balanced state sum, independently introduced in [4] as a cutoff path integral for QGR on the fixed triangulation Δ, with normalization factor (undetermined in [4]) Θ. The definition of the state sum over Δ, in fact, depends only on the elements of the 2-complex $\sigma(\Delta)$ formed by the 2-skeleton of the cellular complex dual to Δ, and it makes sense for any 2-complex with 5-valent vertices and 4-valent edges σ.

The relation with Einstein's theory can be viewed in a number of alternative ways. First, we can directly interpret a colored triangulation as a discretized 4-geometry, where the (scalar Casimir of the) representation ρ_f gives the area of the face f; it can then be shown that the amplitude I associated to a 4-simplex is strictly related to the exponential of the Einstein-Hilbert action of the 4-simplex [21]. We can thus interpret the product of the I's over the 4-simplices as an approximation of the exponential of the action of the triangulated manifold.

Second, the balanced state sum model is a modification of BF theory: the modification implements the constraint that transforms BF theory into GR. The BF action is $S_{\rm BF} = \int \mathrm{Tr}(B \wedge F)$. The GR action can be written as $S_{\rm GR} = \int \mathrm{Tr}(e \wedge e \wedge F)$. The constraint has therefore the structure $B = e \wedge e$. This can equivalently be expressed in the form $B \wedge B = 0$ [22]. In the quantization, B becomes the generator of the representation, and $B \wedge B$ becomes the pseudoscalar Casimir. The balanced representations are precisely the ones in which this Casimir vanishes. Therefore the R projector in (1), which restricts the Fourier components of the field to the balanced ones only, transforms the action (2) (which, without R, is dual to quantum BF theory) into an action dual to QGR (see [13,16,23]).

Finally, the balanced state sum was first obtained as a sum over the *eigenstates* of quantities describing the geometry of a triangulated manifold [7]. The balanced constraint implements the geometrical character of the dynamical variables.

The Feynman expansion (3) of an n-point function of the GFT, gives a function of the boundary labels. These functions can be interpreted as 3-geometry to 3-geometry transition amplitudes, computed to a certain order in a perturbative expansion. They are the (in principle) observable quantities of a quantum theory of gravity [9].

The functions Θ and I in (3) were shown to be bounded in [4] and [20]. Therefore convergence is a question of decay at infinity. We assume the 2-complex to be nondegenerate: each face is bounded by at least three distinct edges and at least three distinct vertices. In the dual picture, "triangulations" in which a 4-simplex is glued to itself, or in which two 4-simplices share more than one tetrahedron, define degenerate 2-complexes. We now sketch the proof of finiteness.

Lemma 1: (Baez-Barrett) Θ and I are bounded. This follows from Theorems 1, 2, and 3 of [20].

Lemma 2: (Baez-Barrett) If $n \geq 3$, the integral

$$J = \int_H dx |K_{\rho_1}(x,x_1)K_{\rho_2}(x,x_2)\cdots K_{\rho_n}(x,x_n)|$$

converges, and for any $0 < \epsilon < 1/3$ there exists a constant $C > 0$ such that for any choice of the points $x_1,\ldots,x_n$,

$$J \leq C\exp\left\{-\frac{n-2-n\epsilon}{n(n-1)}\sum_{i<j} r_{ij}\right\},$$

where r_{ij} is the hyperbolic distance $d(x_i,x_j)$ between x_i and x_j. Using these, one can prove

Lemma 3: For any subset of κ elements $\rho_1\ldots\rho_\kappa$ out of the corresponding four representations appearing in Θ

$$|\Theta| \leq C_\kappa\left(\prod_{i=1}^{\kappa}\rho_i\right)^{-\alpha_\kappa}, \quad \text{where } \alpha_\kappa = \begin{cases} 1 & \text{for } \kappa \leq 3\,, \\ \frac{3}{4} & \text{for } k = 4\,, \end{cases}$$

for some positive constant C_κ.

Lemma 4: For any subset of κ elements $\rho_1\ldots\rho_\kappa$ out of the corresponding ten representations in I, $|I| \leq K_\kappa(\prod_{i=1}^{\kappa}\rho_i)^{-3/10}$ for some positive constant K_κ. Finally

Theorem: *The state sum $\mathcal{A}(\sigma)$ (3) converges for any nonsingular σ.*

Proof: Divide each integration region $\mathbb{R}^+$ into the intervals $[0,1)$, and $[1,\infty)$. The multiple integral decomposes in a finite sum of integrations of three types: (i) All integrations are in $[0,1)$. This term in the sum is finite by Lemma 1. (ii) All the integrations are in the range $[1,\infty)$. This term T is finite since using Lemmas 3 and 4 for $\kappa = 4$ and $\kappa = 10$, respectively, we have

$$T \leq \prod_f \int_1^\infty d\rho_f\,\rho_f^{2-(3/4)n_e-(3/10)n_v}$$
$$\leq \left(\int_1^\infty d\rho\,\rho^{-46/40}\right)^F < \infty\,,$$

where F is the number of faces and n_e (n_v) of edges (vertices) bounding the face f. (iii) m integrations in $[0,1)$, and $F-m$ in $[1,\infty)$. In this case the integral can be bound using Lemmas 3 and 4 as before. The idea is to choose the appropriate subset of representations in the bounds (and the corresponding values of κ) so that only the $m-F$

representations integrated over $[1, \infty)$ appear in the corresponding denominators. Since this is clearly possible, the terms are all finite. We have bound $\mathcal{A}$ by a finite sum of finite terms, QED.

The suggestion that quantum geometry is in some sense discrete is an idea which can be traced all the way to Einstein. Here, the idea is realized in the elegant algebraic categorical form of a state sum, close to TQFT, with a suggestive structural similarity with the Feynman vacuum [24]. The model appears to be the covariant version of loop quantum gravity [1]: a spin foam is a history of spin-networks (the basis states in loop quantum gravity). The suggestion that the covariant form of loop quantum gravity should take the form of a spin foam model has been repeatedly put forward [11–13]. A spin foam represents 4D spacetime; its boundary is a spin-network, which represents 3-space in loop quantum gravity. The n-point functions of the GFT have thus the form of transition amplitudes between spin-network states [9,25]. It is natural to see the two approaches as tentative formulations of one same theory [12].

Several issues are open. First, to understand in which regimes, if any, the expansion in λ is reliable. Quantum self-censorship [24] might play a role in ensuring convergence or asymptotic convergence: the intuition is that any new information in a sufficiently large triangulation would fall into its Schwarzschild radius, and not affect observable quantities. We expect singular 2-complexes to converge as well, but the proof is to be completed. In viewing the model as a Feynman sum over geometries, it is not clear to us how gauge invariance is resolved in the definition of the path integral. But notice: an infinite gauge group volume has in fact already been discarded in defining the integrals; spin foams are intrinsically defined, and naturally diff-invariant; we expect the sum to diverge, and this divergence may include any integration over an infinite gauge group volume (this happens in the GFT formulation of BF). Another question to be investigated is the finiteness of the variant of the model in [26], where timelike as well as spacelike balanced representations are used, and the discreteness of the canonical theory [2] reappears. The results presented here emerge from comparing two distinct ways of viewing spin foam models: the quantum geometric and field theoretic ones. We expect comparison with techniques and results from the other approaches to spin foams, such as the ones in [27], to be productive as well. Finally, a key issue is whether a limit of the theory is indeed GR. Elements of evidence for this exist, as mentioned, but indications of potential difficulties have been pointed out as well [28].

The core of the difficulty in quantum gravity, we think, is to understand the structure of a nontrivial background independent quantum field theory. The present theory might provide an example for such a structure.

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[1] C. Rovelli, Living Rev. Relativity 1 (1998); C. Rovelli and L. Smolin, Phys. Rev. Lett. **61**, 1155 (1988); Nucl. Phys. **B331**, 80 (1990).
[2] C. Rovelli and L. Smolin, Nucl. Phys. **B442**, 593–622 (1995); **B456**, 753 (1995); A. Ashtekar and J. Lewandowski, Classical Quantum Gravity **14**, A55–A82 (1997).
[3] A. Perez and C. Rovelli, Phys. Rev. D **63**, 041501 (2001).
[4] J. Barrett and L. Crane, Classical Quantum Gravity **17**, 3101 (2000).
[5] R. De Pietri, L. Freidel, K. Krasnov, and C. Rovelli, Nucl. Phys. **B574**, 785 (2000).
[6] M. Reisenberger and C. Rovelli, gr-qc/0002083; Classical Quantum Gravity **18**, 121 (2001).
[7] J. Barrett and L. Crane, J. Math. Phys. (N.Y.) **39**, 3296 (1998).
[8] C. Misner, Rev. Mod. Phys. **29**, 497 (1957); S. W. Hawking, in *General Relativity: An Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979).
[9] A. Perez and C. Rovelli, gr-qc/0104034.
[10] E. Brézin, C. Itzykson, G. Parisi, and J. Zuber, Commun. Math Phys. **59**, 35 (1978); F. David, Nucl. Phys. **B257**, 45 (1985); J. Ambjorn, B. Durhuus, and J. Frolich, Nucl. Phys. **B257**, 433 (1985); V. A. Kazakov, I. K. Kostov, and A. A. Migdal, Phys. Lett. **157**, 295 (1985); M. Douglas and S. Shenker, Nucl. Phys. **B335**, 635 (1990); D. Gross and A. Migdal, Phys. Rev. Lett. **64**, 63 (1990).
[11] M. Reisenberger, gr-qc/9412035; J. Iwasaki, J. Math. Phys. (N.Y.) **36**, 6288 (1995).
[12] M. Reisenberger and C. Rovelli, Phys. Rev. D **56**, 3490–3508 (1997); C. Rovelli, Nucl. Phys. **B57**, 28–43 (1997); Phys. Rev. D **59**, 104015 (1999).
[13] J. Baez, Classical Quantum Gravity **15**, 1827–1858 (1998).
[14] D. V. Boulatov, Mod. Phys. Lett. **A7**, 1629 (1992); H. Ooguri, Mod. Phys. Lett. **A7**, 2799 (1992).
[15] V. Turaev and O. Viro, Topology **31**, 865–902 (1992).
[16] A. Perez and C. Rovelli, Nucl. Phys. **B599**, 255 (2001).
[17] D. Oriti and R. Williams, Phys. Rev. D **63**, 024022 (2001).
[18] A. Perez, Nucl. Phys. **B599**, 427 (2001).
[19] E. Buffenoir and P. Roche, J. Math. Phys. (N.Y.) **41**, 7715 (2000).
[20] J. C. Baez and J. W. Barrett, gr-qc/0101107.
[21] L. Crane and D. Yetter, gr-qc/9712087; J. W. Barrett, Adv. Theor. Math. Phys. **2**, 593–600 (1998); J. W. Barrett and R. M. Williams, Adv. Theor. Math. Phys. **3**, 209–214 (1999); L. Freidel and K. Krasnov, J. Math. Phys. (N.Y.) **41**, 1681–1690 (2000).
[22] J. F. Plebanski, J. Math. Phys. (N.Y.) **18**, 2511 (1977); R. Capovilla, T. Jacobson, and J. Dell, Classical Quantum Gravity **8**, 59 (1991).
[23] A. Perez, Ph.D. thesis, University of Pittsburgh, 2001.
[24] L. Crane, gr-qc/0004043.
[25] A. Mikovic, Classical Quantum Gravity **18**, 2827–2850 (2001).
[26] A. Perez and C. Rovelli, gr-qc/0011037.
[27] M. Reisenberger, gr-qc/9711052; L. Freidel and K. Krasnov, Adv. Theor. Math. Phys. **2**, 1183 (1999); J. Iwasaki, gr-qc/0006088.
[28] J. W. Barrett, M. Roček, and R. M. Williams, Classical Quantum Gravity **16**, 1373–1376 (1999); D. Oriti, gr-qc/0106091; M. Reisenberger, J. Math. Phys. (N.Y.) **40**, 2046 (1999); gr-qc/9804061.