Dynamical Foundations of Nonextensive Statistical Mechanics

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We construct classes of stochastic differential equations with fluctuating friction forces that generate a dynamics correctly described by Tsallis statistics. These systems generalize the way in which ordinary Langevin equations underlie ordinary statistical mechanics to the more general nonextensive case. As a main example, we construct a dynamical model of velocity fluctuations in a turbulent flow, which generates probability densities that very well fit experimentally measured probability densities in Eulerian and Lagrangian turbulence. Our approach provides a dynamical reason why many physical systems with fluctuations in temperature or energy dissipation rate are correctly described by Tsallis statistics.

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Recently, there has been considerable interest in the formalism of nonextensive statistical mechanics (NESM) as introduced by Tsallis [1] and further developed by many others (e.g. [2–4]). In the meantime there is growing evidence that the formalism, rather than being just a theoretical construction, is of relevance to many complex physical systems. Applications in various areas have been reported, mainly for systems with either long-range interactions [5–8], multifractal behavior [9,10], or fluctuations of temperature or energy dissipation rate [11–15]. A recent interesting application of the formalism is that to fully developed turbulence [10,12,13]. Precision measurements of probability density functions (pdfs) of longitudinal velocity differences in high-Reynolds number turbulent Couette-Taylor flows are found to agree quite perfectly with analytic formulas of pdfs as predicted by NESM [13]. Despite this apparent success of the nonextensive approach, still the question remains *why* in many cases (such as the above turbulent flow) NESM works so well. In this Letter, we will introduce novel types of stochastic differential equations (SDEs) with three important properties. First, it can be rigorously proven that these generate Tsallis statistics. Second, they are simple extensions of ordinary Langevin equations, thus physically relevant. Third, and most important, the non-Gaussian stationary probability densities agree quite precisely with what is measured in turbulence experiments. Thus, these SDEs yield a dynamical foundation for recently used successful fits in turbulence [13].

To start with, let us first go back to ordinary statistical mechanics and just consider a very simple well-known example, the Brownian particle [16]. Its velocity u satisfies the linear Langevin equation,

$$\dot{u} = -\gamma u + \sigma L(t), \tag{1}$$

where $L(t)$ is Gaussian white noise, $\gamma > 0$ is a friction constant, and σ describes the strength of the noise. The stationary probability density of u is Gaussian with average 0 and variance β^{-1}, where $\beta = \frac{\gamma}{\sigma^2}$ can be identified with the inverse temperature of ordinary statistical mechanics (we assume that the Brownian particle has mass 1).

The above simple situation completely changes if one allows the parameters γ and σ in the SDE to fluctuate as well. To be specific, let us assume that either γ or σ, or both, fluctuate in such a way that $\beta = \gamma/\sigma^2$ is χ^2 distributed with degree n. This means the probability density of β is given by

$$f(\beta) = \frac{1}{\Gamma(\frac{n}{2})}\left\{\frac{n}{2\beta_0}\right\}^{n/2} \beta^{n/2-1} \exp\left\{-\frac{n\beta}{2\beta_0}\right\}. \tag{2}$$

The χ^2 distribution (also called Γ distribution) is a typical distribution that naturally arises in many circumstances. For example, consider n independent Gaussian random variables $X_i, i = 1, \dots, n$ with average 0. If β is given by the sum

$$\beta := \sum_{i=1}^{n} X_i^2, \tag{3}$$

then it has the pdf (2). The average is given by

$$\langle \beta \rangle = n\langle X^2 \rangle = \int_0^\infty \beta f(\beta)\, d\beta = \beta_0, \tag{4}$$

and the variance by

$$\langle \beta^2 \rangle - \beta_0^2 = \frac{2}{n}\beta_0^2 \tag{5}$$

(see also [11]).

Now assume that the time scale on which β fluctuates is much larger than the typical time scale of order γ^{-1} that the Langevin system (1) needs to reach equilibrium. In this case, one obtains for the conditional probability $p(u|\beta)$ (i.e., the probability of u given some value of β),

$$p(u|\beta) = \sqrt{\frac{\beta}{2\pi}} \exp\left\{-\frac{1}{2}\beta u^2\right\}, \tag{6}$$

for the joint probability $p(u,\beta)$ (i.e., the probability to observe both a certain value of u and a certain value of β),

$$p(u,\beta) = p(u|\beta)f(\beta), \tag{7}$$

and for the marginal probability $p(u)$ (i.e., the probability to observe a certain value of u no matter what β is),

$$p(u) = \int p(u|\beta)f(\beta)\,d\beta. \tag{8}$$

The integral (8) is easily evaluated and one obtains

$$p(u) = \frac{\Gamma(\frac{n}{2}+\frac{1}{2})}{\Gamma(\frac{n}{2})}\left(\frac{\beta_0}{\pi n}\right)^{1/2}\frac{1}{(1+\frac{\beta_0}{n}u^2)^{(n/2)+(1/2)}}. \tag{9}$$

Hence, the SDE (1) with χ^2-distributed $\beta = \gamma/\sigma^2$ generates the generalized canonical distributions of NESM [1],

$$p(u) \sim \frac{1}{[1+\frac{1}{2}\tilde{\beta}(q-1)u^2]^{1/(q-1)}}, \tag{10}$$

provided the following identifications are made.

$$\frac{1}{q-1} = \frac{n}{2} + \frac{1}{2} \Longleftrightarrow q = 1 + \frac{2}{n+1}, \tag{11}$$

$$\frac{1}{2}(q-1)\tilde{\beta} = \frac{\beta_0}{n} \Longleftrightarrow \tilde{\beta} = \frac{2}{3-q}\beta_0. \tag{12}$$

We already see from this simple example that the physical inverse temperature $\beta_0 = \langle\beta\rangle$ does not necessarily coincide with the inverse temperature parameter $\tilde{\beta}$ used in the nonextensive formalism (see also [3] for related results). Moreover, Eq. (11) implies $q \geq 1$. Though Wilk *et al.* have shown that the case $q < 1$ can also be associated with fluctuations [17], this case requires more complicated u-dependent distribution functions $f(\beta,u)$, which violate our assumption of time scale separation.

More generally, we may also consider nonlinear Langevin equations of the form

$$\dot{u} = -\gamma F(u) + \sigma L(t), \tag{13}$$

where $F(u) = -\frac{\partial}{\partial u}V(u)$ is a nonlinear forcing. To be specific, let us assume that $V(u) = C|u|^{2\alpha}$ is a power-law potential. The SDE (13) then generates the conditional pdf

$$p(u|\beta) = \frac{\alpha}{\Gamma(\frac{1}{2\alpha})}(C\beta)^{1/(2\alpha)}\exp\{-\beta C|u|^{2\alpha}\}, \tag{14}$$

and for the marginal distributions $p(u) = \int p(u|\beta)f(\beta)\,d\beta$, we obtain, after a short calculation,

$$p(u) = \frac{1}{Z_q}\frac{1}{(1+(q-1)\tilde{\beta}C|u|^{2\alpha})^{1/(q-1)}}, \tag{15}$$

where

$$Z_q^{-1} = \alpha[C(q-1)\tilde{\beta}]^{1/(2\alpha)}\frac{\Gamma(\frac{1}{q-1})}{\Gamma(\frac{1}{2\alpha})\Gamma(\frac{1}{q-1}-\frac{1}{2\alpha})}, \tag{16}$$

and

$$q = 1 + \frac{2\alpha}{\alpha n + 1}, \tag{17}$$

$$\tilde{\beta} = \frac{2\alpha}{1+2\alpha-q}\beta_0. \tag{18}$$

To generalize to N particles in d space dimensions, we may consider coupled systems of SDEs with fluctuating friction forces, as given by

$$\dot{\vec{u}}_i = -\gamma_i\vec{F}_i(\vec{u}_1,\ldots,\vec{u}_N) + \sigma_i\vec{L}_i(t) \qquad i = 1,\ldots,N. \tag{19}$$

Suppose that a potential $V(\vec{u}_1,\ldots,\vec{u}_N)$ exists for this problem such that $\vec{F}_i = [\partial/(\partial\vec{u}_i)]V$. Moreover, assume that the partition function is of the form

$$Z(\beta) = \int d\vec{u}_1\cdots d\vec{u}_N\, e^{-\beta V} \sim \beta^x e^{-\beta y}. \tag{20}$$

If all $\beta_i = \gamma_i/\sigma_i^2$ are given by the same fluctuating χ^2-distributed random variable $\beta_i = \beta$, one ends up with marginal distributions of the form

$$p(\vec{u}_1,\ldots,\vec{u}_N) \sim \frac{1}{[1+\tilde{\beta}(q-1)V(\vec{u}_1,\ldots,\vec{u}_N)]^{1/(q-1)}}, \tag{21}$$

i.e., the generalized canonical distributions of NESM with

$$q = 1 + \frac{2}{n-2x}, \tag{20}$$

and

$$\tilde{\beta} = \frac{\beta_0}{1+(q-1)(x-\beta_0 y)}. \tag{23}$$

However, in many physical applications, the various particles will be dilute and only weakly interacting. Hence, in this case β is expected to fluctuate spatially in such a way that the local inverse temperature β_i surrounding one particle i is almost independent from the local β_j surrounding another particle j. Moreover, the potential is approximately just a sum of single-particle potentials $V(\vec{u}_1,\ldots,\vec{u}_N) = \sum_{i=1}^N V_s(\vec{u}_i)$. In this case integration over all β_i leads to marginal densities of the form

$$p(\vec{u}_1,\ldots,\vec{u}_N) \sim \prod_{i=1}^N \frac{1}{[1+\tilde{\beta}(q-1)V_s(\vec{u}_i)]^{1/(q-1)}}, \tag{24}$$

i.e., the N-particle nonextensive system reduces to products of one-particle nonextensive systems (this type of factorization was, e.g., successfully used in [15]). The truth of what the correct nonextensive thermodynamic description is will often lie between the two extreme cases (21) and (24).

Let us now come to our main physical example, namely, fully developed turbulence. Let u in Eq. (13) represent a local velocity difference in a fully developed turbulent flow as measured on a certain scale r. The parameter $\beta := \gamma/\sigma^2$ of the SDE is an unknown function of the energy dissipation in the flow. Let us consider a model, where

$$\beta = \frac{\gamma}{\sigma^2} = \epsilon_r \tau \Lambda . \tag{25}$$

Here ϵ_r is the (fluctuating) energy dissipation rate averaged over r^3, and τ is a typical time scale during which energy is transferred. Λ is a constant with dimension length^4/time^4; its value is irrelevant for the following. Both ϵ_r *and* τ can fluctuate, and we assume that $\beta \sim \epsilon_r \tau$ is χ^2 distributed. For power-law friction forces, the SDE (13) generates the stationary pdf (15). In Fig. 1 this theoretical distribution is compared with experimental measurements in two turbulence experiments, performed on two very different scales. All distributions have been rescaled to variance 1. Apparently, there is very good coincidence between experimental and theoretical curves, thus indicating that our simple model assumptions are a good approximation of the true turbulent statistics.

Generally, for a turbulent flow the averaged energy dissipation rate $\epsilon_r(\vec{x}, t)$ in a volume V of size r^3 is defined as

$$\epsilon_r(\vec{x}, t) = \frac{1}{r^3} \int_V \epsilon(\vec{x} + \vec{r}\,', t)\, d^3 r', \tag{26}$$

where

$$\epsilon(\vec{x}, t) = \frac{1}{2}\, \nu \sum_{i,j} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2 \tag{27}$$

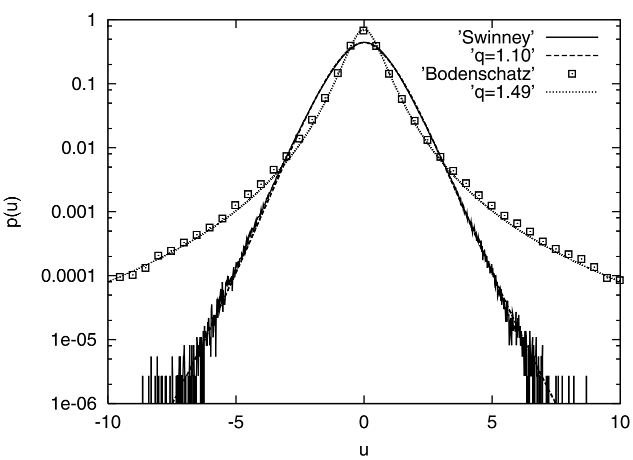

FIG. 1. Histogram of longitudinal velocity differences as measured by Swinney *et al.* [13,18] in a turbulent Couette-Taylor flow with Reynolds number $R_\lambda = 262$ at scale $r = 116\eta$ (solid line), where η is the Kolmogorov length. The experimental data are very well fitted by the analytic formula (15) with $q = 1.10$ and $\alpha = 0.90$ (dashed line). The square data points are a histogram of the acceleration (= velocity difference on a very small time scale) of a Lagrangian test particle as measured by Bodenschatz *et al.* for $R_\lambda = 200$ [19]. These data are well fitted by (15) with $q = 1.49$ and $\alpha = 0.92$ (dotted line).

is the instantaneous dissipation, ν is the kinematic viscosity, and v_i is the velocity fluctuation in i direction. The physical idea of considering the SDE (13) with $\beta \sim \epsilon_r \tau$ is that a test particle in the turbulent flow moves for a while in a certain region with a given ϵ_r, then moves to another region with another ϵ_r, and so on. The average strengths of the effective friction and chaotic driving forces vary in the various regions of the liquid and are effectively modeled by the fluctuating β.

The larger the volume V over which ϵ is averaged, the more independent events will contribute to the fluctuations of $\beta \sim \epsilon_r \tau$. Hence n, according to Eq. (3), is expected to be larger at larger scales r, which means according to Eq. (17) that q is closer to 1. This is indeed confirmed by turbulence experiments, where $q(r)$ is observed to decrease with increasing r in a monotonous way [see [13] for precision measurements of $q(r)$].

At the smallest scale, the Kolmogorov length scale $\eta = (\nu^3/\epsilon)^{1/4}$, one has

$$\beta = \epsilon \tau_\eta \Lambda = (\epsilon \nu)^{1/2} \Lambda = (u_\eta)^2 \Lambda , \tag{28}$$

where $\tau_\eta := (\nu/\epsilon)^{1/2}$ denotes the Kolmogorov time and $u_\eta := \eta/\tau_\eta = (\nu\epsilon)^{1/4}$ is the Kolmogorov velocity (see, e.g., [20]). If ϵ fluctuates, then all these quantities fluctuate as well. It is now reasonable to assume that there are three independently fluctuating Kolmogorov velocities u_η^i, $i = 1, 2, 3$ such that $u_\eta = |\vec{u}_\eta| = \sqrt{\sum_{i=1}^3 (u_\eta^i)^2}$. The three components u_η^i describe the flow of energy into the three different space directions. The simplest model assumption is that these Kolmogorov velocities are Gaussian with average zero. This means we identify $X_i = \sqrt{\Lambda}\, u_\eta^i$ in Eq. (3). Hence at the smallest scale the three space dimensions lead to $n = 3$ or, using Eq. (17), $q \approx \frac{3}{2}$ if $\alpha \approx 1$. This is indeed confirmed by the fit of the small-scale data of the Bodenschatz group in Fig. 1, yielding $q = 1.49$ and an α as given by Eq. (17).

It is interesting to see that this approach allows us to view the fluctuations of ϵ at the Kolmogorov scale in terms of a (hypothetical) ordinary Brownian particle of mass M that is subjected to ordinary thermal noise of temperature T. Its fluctuating velocity $\vec{V}$ coincides with the fluctuating vector $\vec{u}_\eta$ of Kolmogorov velocities. This (constructed) Brownian particle just absorbs the turbulent energy flow at the Kolmogorov scale. It bridges the gap between thermal and macroscopic description. According to the standard Ornstein-Uhlenbeck theory, equipartition of energy yields

$$\tfrac{1}{2} M \langle (u_\eta^1)^2 \rangle = \tfrac{1}{2} kT , \tag{29}$$

which, using Eq. (28), can be written as

$$M = \frac{3kT}{\nu^{1/2} \langle \epsilon^{1/2} \rangle} . \tag{30}$$

In conclusion, our approach yields a dynamical reason for the validity of Tsallis statistics for systems with a fluctuating energy dissipation rate (or, in general, fluctuating friction forces). On the smallest scales of turbulent systems, a prediction for the entropic index q is given, and the obtained probability densities very well agree with recent experimental precision measurements.

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