Transmission of a Cartesian Frame by a Quantum System

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A single quantum system, such as a hydrogen atom, can transmit a Cartesian coordinate frame (three axes). For this it has to be prepared in a superposition of states belonging to different irreducible representations of the rotation group. The algorithm for decoding such a state is presented, and the fidelity of transmission is evaluated.

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There has recently been considerable progress in devising ways to indicate a spatial direction by means of quantum particles. This type of information cannot be represented by a sequence of symbols like 0 and 1, unless the emitter (Alice) and the receiver (Bob) have prearranged a common coordinate system for specifying the numerical values of relevant angles. Physical objects have to be sent. Preceding works [1–4] have considered the use of spins for transmitting a single direction. The simplest method [1] is to send these spins polarized along the direction that one wishes to indicate. This, however, is not the most efficient procedure: when two spins are transmitted, a higher accuracy is achieved by preparing them with *opposite* polarizations [2]. If there are more than two spins, optimal results are obtained with *entangled* states [3,4].

This Letter presents a method for the transmission of a complete Cartesian frame. If many spins are available, a simple possibility would be for Alice to use half of them for indicating her x axis and the other half for her y axis. However, the two directions found by Bob may not then be exactly perpendicular, because separate transmissions have independent errors due to limited angular resolution. Some adjustment will be needed to obtain Bob's best estimates for the x and y axes, before he can infer from them his guess of Alice's z direction. This method is not optimal, and it is obviously not possible to proceed in this way if a *single* quantum messenger is available. Here we shall show how a single hydrogen atom (formally, a spinless particle in a Coulomb potential) can transmit a complete frame [5].

Consider the nth energy level of that atom (a Rydberg state). Its degeneracy is $d = n^2$ because the total angular momentum may take values $j = 0, \dots, n-1$, and for each one of them $m = -j, \dots, j$. Alice indicates her xyz axes by sending the atom in a state

$$|A\rangle = \sum_{j=0}^{n-1} \sum_{m=-j}^{j} a_{jm} |j, m\rangle, \qquad (1)$$

with normalized coefficients a_{jm} that will be specified below. Bob then performs a covariant measurement [6] in order to evaluate the Euler angles $\psi\theta\phi$ that would rotate his own xyz axes into a position parallel to Alice's axes. Bob's detectors (ideally, there is an infinite number of them [7]) have labels $\psi\theta\phi$ and the mathematical representation of his apparatus is a *positive operator valued measure* (POVM) [8,9], namely a resolution of identity by a set of positive operators:

$$\int d_{\psi\theta\phi} E(\psi\theta\phi) = \mathbf{1}, \qquad (2)$$

where $d_{\psi\theta\phi} \equiv \sin\theta\, d\psi\, d\theta\, d\phi / 8\pi^2$ is the SO(3) Haar measure for Euler angles [10], and $E(\psi\theta\phi) = |\psi\theta\phi\rangle\langle\psi\theta\phi|$. The vectors $|\psi\theta\phi\rangle$ will be specified below. The probability that the detector labeled $\psi\theta\phi$ is excited is given by

$$P(\psi\theta\phi) = \langle A| d_{\psi\theta\phi} E(\psi\theta\phi) |A\rangle = d_{\psi\theta\phi} |\langle A \mid \psi\theta\phi\rangle|^2. \qquad (3)$$

Our task is to construct vectors $|\psi\theta\phi\rangle$ such that Eq. (2) is satisfied (that is, the probabilities sum up to one) and Bob's expected error is minimal.

Following the method of Ref. [4], we define a fiducial vector for Bob,

$$|B\rangle = \sum_{j=0}^{n-1} \sqrt{2j+1} \sum_{m=-j}^{j} b_{jm} |j, m\rangle, \qquad (4)$$

where the coefficients b_{jm} are normalized for each j separately:

$$\sum_{m=-j}^{j} |b_{jm}|^2 = 1 \quad \forall\, j. \qquad (5)$$

In Ref. [4] a single value of m was used; here we need all the values. Note that Eq. (1) was written with Alice's notations (m is the angular momentum along her z axis), while Eq. (4) is written with Bob's notations (m refers to his z axis). This issue will be dealt with later.

We now define

$$|\psi\theta\phi\rangle = U(\psi\theta\phi) |B\rangle, \qquad (6)$$

where $U(\psi\theta\phi)$ is the unitary operator for a rotation by Euler angles $\psi\theta\phi$. Note that since $|B\rangle$ is a direct sum of vectors, one for each value of j, then likewise $U(\psi\theta\phi)$ is a direct sum with one term for each irreducible representation,

$$U(\psi\theta\phi) = \sum_j \oplus \mathcal{D}^{(j)}(\psi\theta\phi), \qquad (7)$$

where the $\mathcal{D}^{(j)}(\psi\theta\phi)$ are the usual irreducible unitary rotation matrices [10]. To prove that Eq. (2) is satisfied, we note that its left-hand side is invariant if multiplied by $U(\mu\nu\rho)$ on the left and $U(\mu\nu\rho)^\dagger$ on the right, for any arbitrary Euler angles $\mu\nu\rho$ (because these unitary matrices represent group elements and therefore have the group multiplication properties) [11]. It then follows from a generalization of Schur's lemma [12] that the left-hand side of (2) is a direct sum of *unit* matrices, owing to the presence of the factor $(2j + 1)$ which is the dimensionality of the corresponding irreducible representation. Therefore Eq. (2) is satisfied.

The detection probability (3) can thus be written as $P(\psi\theta\phi) = d_{\psi\theta\phi}|\langle A|U(\psi\theta\phi)|B\rangle|^2$. To compute this expression explicitly, we must use a uniform system of notations for $|A\rangle$ and $|B\rangle$—recall that Eq. (1) was written in Alice's basis, and Eq. (4) in Bob's basis. It is easier to rewrite Alice's vector $|A\rangle$ in Bob's language. For this we have to introduce the Euler angles $\xi\eta\zeta$ that rotate Bob's xyz axes into Alice's axes (that is, $\xi\eta\zeta$ are the true, but unknown values of the angles $\psi\theta\phi$ sought by Bob). The unitary matrix $U(\xi\eta\zeta)$ represents an *active* transformation of Bob's vectors into Alice's. Therefore, $U(\xi\eta\zeta)^\dagger$ is the *passive* transformation (p. 216 in Ref. [8]) from Bob's notations to those of Alice, and $U(\xi\eta\zeta)$ is the corresponding transformation from Alice's notations to Bob's. Written in Bob's notations, Alice's vector $|A\rangle$ becomes $U(\xi\eta\zeta)\,|A\rangle$ so that, in Eq. (3), $\langle A|$ becomes $\langle A|U(\xi\eta\zeta)^\dagger$. Let us therefore define

$$U(\alpha\beta\gamma) = U(\xi\eta\zeta)^\dagger U(\psi\theta\phi). \tag{8}$$

The Euler angles $\alpha\beta\gamma$ have the effect of rotating Bob's Cartesian frame into his *estimate* of Alice's frame, and then rotating back the result by the *true* rotation from Alice's to Bob's frame. That is, the angles $\alpha\beta\gamma$ indicate Bob's measurement error, and the probability of that error is

$$P(\alpha\beta\gamma) = d_{\alpha\beta\gamma}|\langle A|U(\alpha\beta\gamma)|B\rangle|^2, \tag{9}$$

where $d_{\alpha\beta\gamma} = \sin\beta\, d\alpha\, d\beta\, d\gamma/8\pi^2$. Note that in the above equation $|A\rangle$ is written with Alice's notations as in (1), and $|B\rangle$ with Bob's notations as in (4).

Of course, Bob cannot know the values of $\alpha\beta\gamma$. His measurement yields only some value for $\psi\theta\phi$. The following calculation that employs $\alpha\beta\gamma$ has the sole purpose of estimating the expected accuracy of the transmission (which does not depend on the result $\psi\theta\phi$).

We must now choose a suitable quantitative criterion for that accuracy. When a single direction is considered, it is convenient to define the error [13] as $\sin(\omega/2)$, where ω is the angle between the true direction and the one estimated by Bob. The mean square error is

$$\langle\sin^2(\omega/2)\rangle = (1 - \langle\cos\omega\rangle)/2 = 1 - F, \tag{10}$$

where F is usually called the *fidelity* [4]. When we consider a Cartesian frame, we likewise define fidelities for each axis. Note that $\cos\omega_k$ (for the kth axis) is given by the corresponding diagonal element of the orthogonal (classical) rotation matrix. Explicitly, we have [14]

$$\cos\omega_z = \cos\beta, \tag{11}$$

and

$$\cos\omega_x + \cos\omega_y = (1 + \cos\beta)\cos(\alpha + \gamma), \tag{12}$$

whence, by Euler's theorem,

$$\cos\omega_x + \cos\omega_y + \cos\omega_z = 1 + 2\cos\Omega, \tag{13}$$

where Ω has a simple physical meaning: it is the angle for carrying one frame into the other by a single rotation.

The expectation values of the above expressions are obtained with the help of Eq. (9):

$$\langle f(\alpha\beta\gamma)\rangle = \int d_{\alpha\beta\gamma}|\langle A|U(\alpha\beta\gamma)|B\rangle|^2 f(\alpha\beta\gamma), \tag{14}$$

where, explicitly,

$$\langle A|U(\alpha\beta\gamma)|B\rangle = \sum_{j,m,r} a^*_{jm} b_{jr} \langle j,m|\mathcal{D}^{(j)}(\alpha\beta\gamma)\,|\,j,r\rangle. \tag{15}$$

The unitary irreducible rotation matrices $\mathcal{D}^{(j)}(\alpha\beta\gamma)$ have components [10]

$$\langle j,m|\mathcal{D}^{(j)}(\alpha\beta\gamma)\,|\,j,r\rangle = e^{i(m\alpha + r\gamma)} d^{(j)}_{mr}(\beta), \tag{16}$$

where the $d^{(j)}_{mr}(\beta)$ can be expressed in terms of Jacobi polynomials. Collecting all these terms, we finally obtain, after many tedious analytical integrations over products of Jacobi polynomials [15],

$$\langle f(\alpha\beta\gamma)\rangle = \sum f_{jkmnrs} a^*_{jm} b_{jr} a_{kn} b^*_{ks}, \tag{17}$$

where the numerical coefficients f_{jkmnrs} depend on our choice of $f(\alpha\beta\gamma)$ in Eqs. (11)–(13). The problem is to optimize the components a_{jm} (normalized to 1), and b_{jm} satisfying the constraints (5), so as to maximize the above expression. For further use, it is convenient to define a matrix

$$M_{jm,kn} = \sum_{r,s} f_{jkmnrs} b_{jr} b^*_{ks}, \tag{18}$$

so that

$$\langle f(\alpha\beta\gamma)\rangle = \sum M_{jm,kn} a^*_{jm} a_{kn} = \langle A|M|A\rangle. \tag{19}$$

First consider a simple case: to transfer only the z axis, we wish to maximize $\langle\cos\beta\rangle$. An explicit calculation yields

$$f_{jkmnrs} = \delta_{mn}\delta_{rs} g_{jk}. \tag{20}$$

The matrix g_{jk} which is defined by the above equation has nonvanishing elements

$$g_{jj} = ns/[j(j+1)], \tag{21}$$

and

$$g_{j,j-1} = g_{j-1,j} = \frac{1}{j}\sqrt{\frac{(j^2 - n^2)(j^2 - s^2)}{4j^2 - 1}}. \qquad (22)$$

The δ_{mn} term in (20) implies that, for any choice of Bob's fiducial vector $|B\rangle$, the matrix M in (19) is block diagonal, with one block for each value of m. The optimization of Alice's signal results from the highest eigenvalue of that matrix. This is the highest eigenvalue of one of the blocks, so that a single value of m is actually needed. A similar (slightly more complicated) argument applies if Alice's vector is given and we optimize Bob's fiducial vector. This result proves the correctness of the intuitive assumption that was made in [3,4] where a single value of m was used. It was then found that when $m = 0$ (the optimal value) the expected error asymptotically behaves as $1.446/d$, where d is the effective number of Hilbert space dimensions. In the present case, $d = (j_{\max} + 1)^2$, which is the degeneracy of the nth energy level.

If we want to transfer two axes, we use Eq. (12) and calculate the matrix elements for $\langle(1 + \cos\beta)\cos(\alpha + \gamma)\rangle$. (It is curious that they are simpler than those for $\langle\cos(\alpha + \gamma)\rangle$ alone.) We obtain

$$f_{jkmnrs} = \delta_{m,n-1}\delta_{r,s-1}h_{jk} + \delta_{n,m-1}\delta_{s,r-1}h_{kj}, \qquad (23)$$

where the nonvanishing elements of h_{jk} are

$$h_{jj} = \frac{[(j - n + 1)(j + n)(j - s + 1)(j + s)]^{1/2}}{2j(j + 1)}, \qquad (24)$$

$$h_{j,j-1} = \frac{[(j - n + 1)(j - n)(j - s + 1)(j - s)]^{1/2}}{2j(4j^2 - 1)^{1/2}}, \qquad (25)$$

$$h_{j-1,j} = \frac{[(j + n - 1)(j + n)(j + s - 1)(j + s)]^{1/2}}{2j(4j^2 - 1)^{1/2}}. \qquad (26)$$

Note that the h_{jk} matrix, whose elements depend on n and s, is not symmetric (while g_{jk} was). This is because it comes from the operator $e^{i(\alpha+\gamma)}$ which is not Hermitian. However, the two terms of (23) together, which corresponds to $\cos(\alpha + \gamma)$, have all the symmetries required by the other terms $a^*_{jm}b_{jr}a_{kn}b^*_{ks}$ in Eq. (17). Finally, if we wish to optimize directly the three Cartesian axes (without losing accuracy by inferring z from the approximate knowledge of x and y) we use all the terms of (13), that is, both those of (20) and of (23).

It now remains to find the vectors $|A\rangle$ and $|B\rangle$ that minimize the transmission error. For small values of j, we used Powell's method [16] without imposing any restrictions on $|A\rangle$ and $|B\rangle$ other than their normalization conditions. As intuitively expected, we found that the optimal vectors satisfy

$$b_{jm} = a_{jm}\left(\sum_n |a_{jn}|^2\right)^{-1/2}, \qquad \forall\, j. \qquad (27)$$

This means that Bob's vector should look as much as possible like Alice's signal, subject to the restrictions imposed by the constraint (5).

Taking this property for granted is the key to a more efficient optimization method, as follows: assume any b_{jm}, so that the bilinear form (18) is known. Find its highest eigenvalue and the corresponding eigenvector a_{jm}. From the latter, get new components b_{jm} by means of (27), and repeat the process until it converges (actually, a few iterations are enough). The results are shown in Fig. 1. It is seen that there is little advantage in optimizing only two axes, if for any reason the third axis is deemed less important. If the three axes are simultaneously optimized, it can be shown [17] that the mean square error tends asymptotically to $1/\sqrt{d}$.

It is not surprising that this result is weaker than the one for a single axis, which was $1.446/d$. The obvious reason is that we are now transmitting a three-dimensional rotation operation that can be applied to any number of directions, not only to three orthogonal axes. Indeed, consider any set of unit vectors $\mathbf{e}^{\mu}_{m}$, where $m = 1, 2, 3$, and μ is a label

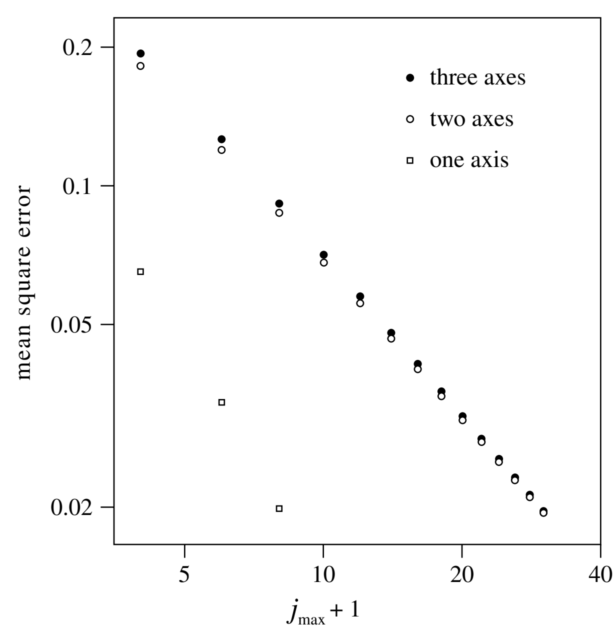

FIG. 1. Mean square error (per axis) for the transmission of the directions of one, two, or three axes, by a single quantum carrier. (This figure was drawn with a log-log scale.)

for identifying the vectors. Let w_μ be a positive weight factor attached to each vector, indicating its importance. Let $R(\alpha\beta\gamma)$ be the classical orthogonal rotation matrix [14] for Euler angles $\alpha\beta\gamma$. Then the cosine of the angle between Bob's estimate of $\mathbf{e}_m^\mu$ and the true direction of that vector is

$$\cos\omega_\mu = \sum_{m,n} R_{mn}(\alpha\beta\gamma)\mathbf{e}_m^\mu \mathbf{e}_n^\mu . \tag{28}$$

With the same notations as before, we have

$$\langle f(\alpha\beta\gamma)\rangle = \sum_\mu w_\mu \langle\cos\omega_\mu\rangle = \sum_{m,n} \langle R_{mn}(\alpha\beta\gamma)\rangle C_{mn} , \tag{29}$$

where

$$C_{mn} = \sum_\mu w_\mu \mathbf{e}_m^\mu \mathbf{e}_n^\mu . \tag{30}$$

This is a positive matrix which depends only on the geometry of the set of vectors whose transmission is requested. We can now diagonalize C_{mn} and write it in terms of three orthogonal vectors, possibly with different weights. Therefore, no essentially new features follow from considering more than three directions.

Finally, we note that all the above calculations, as well as those in preceding works [1–4], assume that Alice and Bob have coordinate frames with the same chirality (this can be checked locally by using weak interactions). If the chiralities are opposite, then all the directions inferred by Bob should be reversed (because directions are polar vectors while spins are axial vectors).

In summary, we have shown that a single structureless quantum system (a point mass in a Coulomb potential) can transfer information on the orientation of a Cartesian coordinate system with arbitrary accuracy. This conclusion is very surprising. No spherically symmetric classical object can achieve this result. Only those having an asymmetric internal structure, such as an asymmetric rigid body, can reliably transmit a Cartesian frame. A spherically symmetric top can at most indicate one direction (that of its angular momentum). This is one more example of the remarkable ability of quantum systems to encode information more efficiently than classical ones.

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[7] In any real experiment, the number of detectors must be finite. It can be shown that a finite POVM with $4n^3$ detectors can give the same angular resolution as obtained here. Instead of continuous integrals, there are $2n$ equidistant values for ψ and for ϕ, and an n-point Gaussian integration is performed over $\cos\theta$. Another method is presented by E. Bagan, M. Baig, and R. Muñoz-Tapia, Phys. Rev. A **64**, 022305 (2001).
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[9] Owing to Gleason's theorem, any detection procedure is mathematically represented by a POVM. However, quantum mechanics is only a theoretical framework and it does not tell how to actually build the corresponding experimental setup. Likewise, Maxwell's equations do not say how to build an electric motor, and thermodynamics does not tell how to build steam engines.
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