# Quantum Phase Transition for $\gamma$-Soft Nuclei

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We examine a quantum phase transition in $\gamma$-soft nuclei, where the O(6) limit is simultaneously a dynamical symmetry of the U(6) group of the interacting boson model and a critical point of a prolate-oblate phase transition. This is the only example of phase transitional behavior that can be described analytically for a finite $s, d$ boson system.

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Recently, the topic of quantum phase transitional behavior of atomic nuclei has received a lot of attention, when it was shown that new symmetries can describe atomic nuclei at the critical points [1,2]. The new symmetries, called $X(5)$ and $E(5)$, are obtained within the framework of the collective model [3] under some simplifying approximations. Remarkably, the parameter-free predictions provided by these symmetries are closely realized in some atomic nuclei [4,5]. The phase transitions considered in these references are those of the ground state deformation, which can conveniently be described by the three Euler angles defining the orientation of the deformed nucleus in space and the deformation parameters $\beta$ and $\gamma$ defining the form of the ellipsoid. These quantum phase transitions take place at zero temperature and depend on the number of nucleons.

The two new critical point symmetries focus on the $\beta$ degree of freedom and are based on a separation of the $\beta$ degree of freedom from the $\gamma$ degree of freedom. An infinite square well potential in the $\beta$ degree of freedom is imposed. Since these descriptions are embedded in the framework of the collective model, they do not incorporate finite $N$ effects reflecting the finite number of nucleons in actual atomic nuclei.

However, finite $N$ effects are important, for at least two reasons. First, classically, phase transitions occur only for systems having an infinite number of constituents and therefore an extension of the concept to systems having a limited number of constituents is of importance. Second, since atomic nuclei contain only an integer number of nucleons, nature does not allow us to vary the control parameters continuously in the region where the phase transition occurs. While the experimental limitation to the integer nucleon number leads to discrete changes in the properties of atomic nuclei around the critical point, theoretical models do allow one to continuously vary the appropriate control parameter. By exploiting this idea, we have recently studied [6] general characteristics of the phase transitional character in collective models.

Using the interacting boson model (IBM) [7], one obtains a very simple two-parameter description leading to a symmetry triangle describing many atomic nuclei. This Casten triangle has the three dynamical symmetries of the IBM: U(5), O(6), and SU(3) in the three vertices. These correspond to vibrational nuclei with a spherical form [U(5)], an axially symmetric prolate rotor with a minimum at $\gamma = 0°$ [SU(3)], and an axially asymmetric rotor with a flat potential in $\gamma$ [O(6)]. Note that, here, we use the convention of Dieperink *et al.* [8] with $\beta \geq 0$ and $0° \leq \gamma \leq 60°$.

In Ref. [6] a close relation was observed between the critical points and the turning points of many observables (although these points do not precisely coincide, presumably due to the finite value of $N$ [6]). The two critical point symmetries occur at the point where the potential becomes deformed, along the U(5)-SU(3) leg of the triangle for $X(5)$ and along the U(5)-O(6) leg for $E(5)$ for $N \rightarrow \infty$. These correspond to first- and second-order phase transitions, respectively, as shown using the coherent state formalism [8,9].

No phase transition is found [9] between the SU(3) and O(6) vertices of the triangle. However, as discussed in Refs. [10,11] in the context of catastrophe theory, an analysis of the separatrix of the IBM-1 Hamiltonian in the coherent state formalism shows that there is a phase transition in between oblate and prolate deformed nuclei. It is the purpose of this Letter to examine this, widely unrecognized, phase transition and its critical point symmetry, which, in fact, *coincides* with the O(6) limit, from the standpoint of physical observables. These observables, in contrast to the quantities studied in Refs. [10,11], do not rely on approximations inherent to the coherent state formalism.

The control variable for changes between SU(3) and O(6) is the parameter $\chi$ in the quadrupole operator:

$$Q = (s^\dagger \tilde{d} + d^\dagger s)^{(2)} + \chi (d^\dagger \tilde{d})^{(2)}. \qquad (1)$$

which traditionally is varied from $\chi = -\sqrt{7}/2$ [SU(3) value] down to $\chi = 0$ [O(6) value]. For $\chi = -\sqrt{7}/2$ the energy functional has a minimum at $\gamma = 0°$, corresponding to a prolate shape.

We first note that the derivatives of several observables peak at the O(6) limit. Calculations with the extended consistent-Q formalism Hamiltonian [12,13] of the IBM-1 are illustrated in Fig. 1 for the first derivatives of $q_2$, $\sigma_\gamma$, and the SU(3) wave-function entropy, where the first two

  

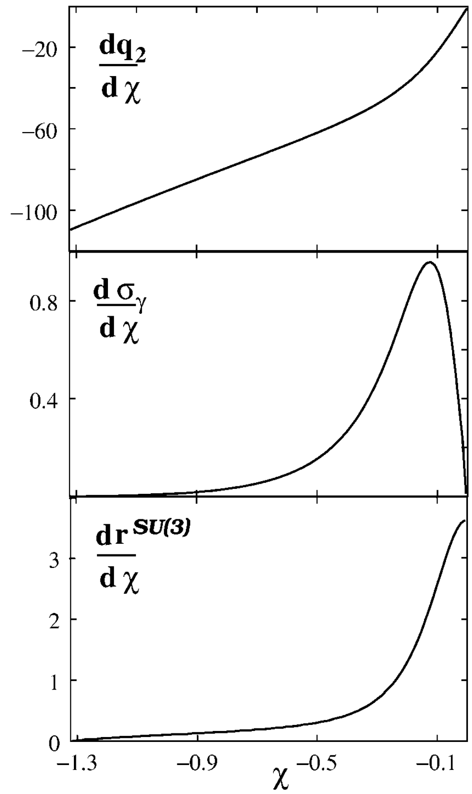


FIG. 1. Behavior of the first derivatives of the $Q$ invariants $q_2$, $\sigma_\gamma$, and the ground state wave-function entropy $r^{\mathrm{SU}(3)}$ in an SU(3) basis when going from SU(3) to O(6), calculated for $N = 10$ bosons. The effective boson charge is kept constant at $e_B = 1$, which affects the absolute scale of $dq_2/d\chi$.

are $Q$ invariants [14–16] representing $\beta^2$ and the softness in $\gamma$, and the last is a measure of the spreading of a wave function in the SU(3) basis [17,18].

This observation leads to the natural issue of whether the O(6) limit itself is not a critical point symmetry of a phase transition. We will show that it is, and that the origin of the phase transition lies in the form of the potential in the $\gamma$ degree of freedom. This is logical because the energy functional has a minimum at an axially symmetric shape in SU(3) but becomes totally flat in $\gamma$ at O(6). Thus the control variable for changes in $V(\gamma)$ is likewise $\chi$. As we deal here with a phase transition in $\gamma$, the derivative of $q_2$ in Fig. 1 does not show a strong peak at the O(6) limit.

In the construction of the symmetry triangle, solutions with positive $\chi$ have only occasionally been considered [10,19], because they correspond to oblate ground state deformations which are seldom found in actual atomic nuclei. However, the interacting boson approximation (IBA) allows solutions with positive $\chi$ as well. The energy functional then has a minimum at $\gamma = 60°$, corresponding to an oblate deformation. The energy functionals as a function of $\gamma$ and for several values of $\chi$ are shown in Fig. 2, using the parametrization of Ref. [20] and for $\xi = 1$, and show a phase transitional behavior at the O(6) value $\chi_{\mathrm{crit}} = 0$. For $\chi < 0$, $V_{\min}$ is always at $\gamma = 0°$, whereas, for any

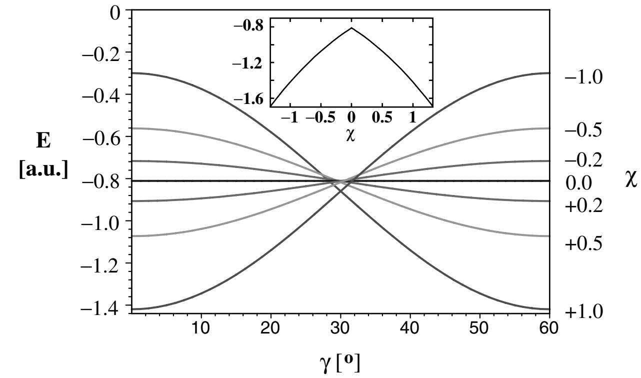


FIG. 2. The $\gamma$ dependence of the IBA energy functional calculated on the SU(3) to O(6) leg of the Casten triangle, for several $\chi$ values, both negative and positive. The calculation was performed for $N = 30$. The inset shows the energy of the minimum as a function of $\chi$.

$\chi > 0$, it is at $\gamma = 60°$. One thus observes a discontinuous jump of the location of the minimum in $\gamma$ at the phase transition. The inset of Fig. 2 shows the energy of the minimum, $E_{\min}(\beta, \gamma)$, as a function of $\chi$. One clearly observes the kink typical for a first-order transition at $\chi = 0$. This phase transition happens for any Hamiltonian yielding a deformed minimum.

By extending the range of $\chi$ to positive values the coincidence of the critical point and the turning point or extremum of the $Q$ invariants $q_2$, $K_3$, and $\sigma_\gamma$ is clearly seen (see Fig. 3). Note that $K_3$ is strongly connected to the $\gamma$ deformation, and here it has a sign and not only an absolute value as in the definition in Ref. [14]. As we pass from O(6) to oblate deformation, the sign of $K_3$ becomes important. The value of $\sigma_\gamma$ has a sharp maximum at O(6).

For the value $\underline{\chi = +}\sqrt{7}/2$, one obtains a dynamical symmetry called $\overline{\mathrm{SU}(3)}$, corresponding to an oblate axially symmetric rotor. By extending the symmetry triangle from $\chi = -\sqrt{7}/2$ to $\chi = +\sqrt{7}/2$ we obtain a variation of the Casten triangle, which is shown in Fig. 4. Here, to the usual Casten triangle, U(5)-SU(3)-O(6), we add a corresponding triangle for U(5)-$\overline{\mathrm{SU}(3)}$-O(6). As all legs in the extended triangle contain a critical point, we also expect a critical point on the transition leg U(5)-$\overline{\mathrm{SU}(3)}$. This critical point has indeed been found using the methods of Ref. [6]. It is natural to speculate that to this critical point there corresponds for $N \to \infty$ also a critical point symmetry (which might be denoted by $\overline{X(5)}$ although a detailed study still needs to be done). The new triangle also maps the phase transitions which occur inside the triangle. They are the first-order transition occurring on the O(6)-U(5) axis to the dot in the center and the first-order transition from thereon outwards to the legs. The central dot itself represents the second order-transition [$E(5)$].

Although oblate nuclei or transitions from $\gamma$-soft to oblate shapes are rare, this situation may change in the future with access to neutron-rich exotic nuclei, where shell reordering may lead to regions of oblate nuclei. Of course,

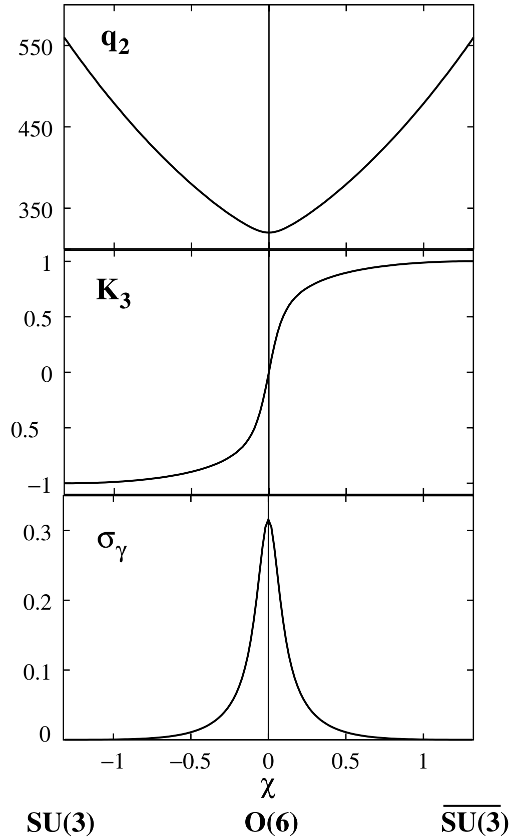


FIG. 3. Behavior of the shape invariants $q_2$, $K_3$ and $\sigma_\gamma$ as a function of $\chi$ for negative and positive values of $\chi$, calculated for $N = 10$ bosons. The observables $q_2$ and $K_3$ are the quadratic and cubic shape invariants, and $\sigma_\gamma$ is an invariant with power six which measures the amount of fluctuations in the variable $\gamma$.

atomic nuclei that are well described by the O(6) limit do occur, for instance, in $^{196}$Pt [21] and nuclei in the Xe-Ba mass region [22]. They are found in mass regions with a relatively small number of valence nucleons. It would be of high interest to search for atomic nuclei having a slightly oblate deformation in more neutron-rich nuclei in the Pt mass region in order to study this possible phase transition.

Returning to the phase transitional behavior at O(6), one notes several important differences with the critical point symmetries $E(5)$ and $X(5)$. First, the phase transition concerns the deformation variable $\gamma$ instead of $\beta$. So, physically, it is a jump from prolate to oblate shapes at the critical point, $\chi = 0$. Second, because the O(6) limit can be exactly solved [7], one does not need any approximation to describe analytically the behavior at the critical point. Finally, and most importantly, the exact solution is obtained for any $N$ and not only for $N = \infty$.

To demonstrate that this is a finite $N$ description of a phase transition, several observables, all related to the $\gamma$ degree of freedom, can be studied at the critical point as a function of $N$ in an exact way. As an example, Fig. 5 shows some typical experimental signatures but now calculated for $N = 16$. Note that $Q(2_1^+)$ and $B(E2; 2_2^+ \rightarrow 2_1^+)$ are given in arbitrary units due to their dependence on an effective boson charge. We see in Fig. 6, using as an example the SU(3) ground state wave-function entropy, how the variation around $\chi = 0$ becomes sharper and sharper as $N$ approaches infinity.

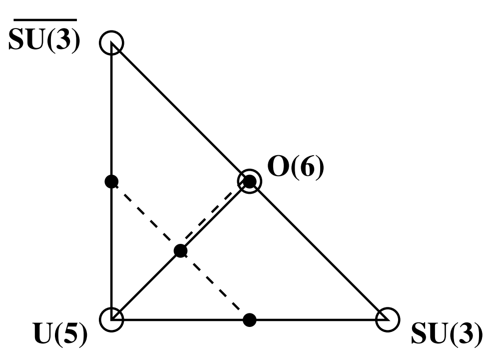


FIG. 4. The Casten triangle, expanded to include the oblate symmetry $\overline{\text{SU(3)}}$. The circles indicate the dynamical symmetries of the IBM model; the solid dots represent critical points along the legs of the triangle. Their specific location follows the convention of Ref. [6]. The dashed lines represent trajectories of critical point behavior inside the triangle. On this modified triangle, O(6) forms both a dynamical symmetry and a critical point for the SU(3) to $\overline{\text{SU(3)}}$ transition.

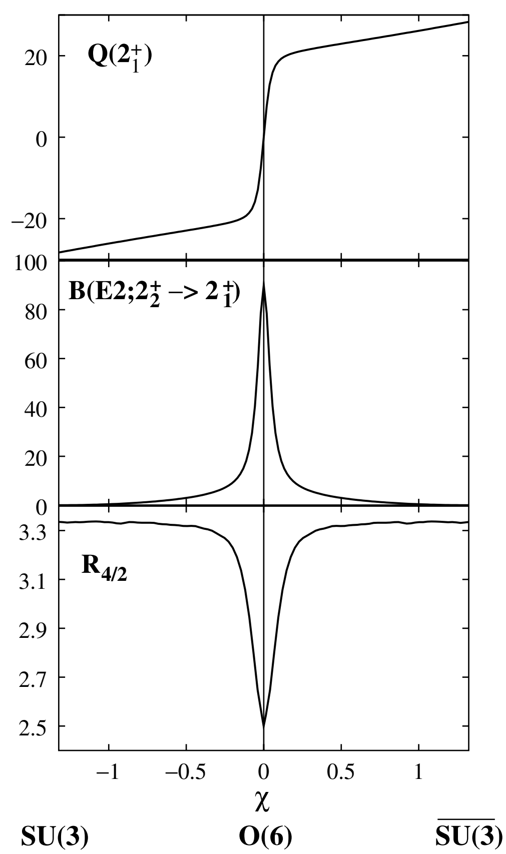


FIG. 5. Experimental quantities illustrating the phase transitional behavior are shown as a function of $\chi$ for $N = 16$. The ratio $R_{4/2}$ is the ratio of the $4_1^+$ and $2_1^+$ energies.

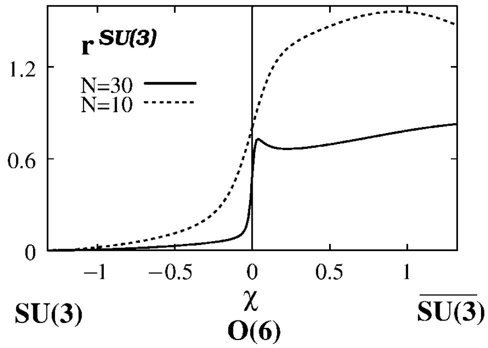


FIG. 6. Wave-function entropy in an SU(3) basis for the ground state for boson numbers $N = 10$ and $N = 30$. The phase transitions become much sharper with increasing boson numbers.

In conclusion, we have discussed the phase transition in $\gamma$-soft nuclei, which occurs because of the discontinuous jump of the location of the minimum of the energy functional in the $\gamma$ degree of freedom. At the O(6) limit the critical behavior can be exactly solved for any number of interacting bosons. Thus, it provides a base to study $N$-dependent effects in quantum phase transitions. The O(6) limit is therefore unique in being, simultaneously, both a dynamical symmetry of the U(6) group of the IBA and a critical point of a prolate-oblate phase transition.

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[1] F. Iachello, Phys. Rev. Lett. **85**, 3580 (2000).
[2] F. Iachello, Phys. Rev. Lett. **87**, 052502 (2001).
[3] A. Bohr and B. R. Mottelson, *Nuclear Structure* (Benjamin, New York, 1975).
[4] R. F. Casten and N. V. Zamfir, Phys. Rev. Lett. **85**, 3584 (2000).
[5] R. F. Casten and N. V. Zamfir, Phys. Rev. Lett. **87**, 052503 (2001).
[6] V. Werner, P. von Brentano, R. F. Casten, and J. Jolie (to be published).
[7] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, Cambridge, UK, 1987).
[8] A. E. L. Dieperink, O. Scholten, and F. Iachello, Phys. Rev. Lett. **44**, 1747 (1980).
[9] J. N. Ginocchio and M. W. Kirson, Phys. Rev. Lett. **44**, 1744 (1980).
[10] E. Lopez-Moreno and O. Castanos, Phys. Rev. C **54**, 2374 (1996).
[11] E. Lopez-Moreno and O. Castanos, Rev. Mex. Fis. **44**, 48 (1998).
[12] D. D. Warner and R. F. Casten, Phys. Rev. C **28**, 1798 (1983).
[13] P. O. Lipas, P. Toivonen, and D. D. Warner, Phys. Lett. **155B**, 295 (1985).
[14] D. Cline, Annu. Rev. Nucl. Part. Sci. **36**, 683 (1986).
[15] K. Kumar, Phys. Rev. Lett. **28**, 249 (1972).
[16] V. Werner *et al.,* Phys. Rev. C **61**, 021301 (2000).
[17] P. Cejnar and J. Jolie, Phys. Lett. B **420**, 241 (1998).
[18] P. Cejnar and J. Jolie, Phys. Rev. E **58**, 387 (1998).
[19] J. Jolie and P. Cejnar, J. Phys. G **25**, 843 (1999).
[20] F. Iachello, N. V. Zamfir, and R. F. Casten, Phys. Rev. Lett. **81**, 1191 (1998).
[21] R. F. Casten and J. A. Cizewski, Nucl. Phys. **A309**, 477 (1978).
[22] R. F. Casten and P. von Brentano, Phys. Lett. **152B**, 22 (1985).