Resonance Absolute Quantum Reflection at Selected Energies

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We present one-dimensional local potentials with an absolute reflection at a given energy value which can be above barriers. The corresponding energy dependence of a reflection coefficient exhibits resonance behavior. The inversion technique provides the potentials with specified widths of reflection resonances, their number and positions. The multichannel systems (exact models) with a complete reflection are also given.

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The foundation of quantum mechanics is, after all, the laws of wave motion in different potentials. The most interesting are the potentials with special qualitative properties because they give a deeper insight into the peculiarities of the microworld. We have revealed the phenomenon of *total reflection* at selected energy points by specific potentials. It is remarkable that this can occur even for the *above-barrier* motion. The possibility of such a resonance reflection has never before been mentioned.

Consider the one-dimensional Schrödinger equation on the whole axis with time-independent local potentials (we put $\hbar^2 = 2m = 1$) . The mathematical aspects of the corresponding scattering problem can be found in [1–4], including the inverse problem. For our purpose, we shall at first need potentials of the Neumann-Wigner–type on the *semiaxis* $0 \leq x < \infty$ which have bound states embedded into the continuum (BSEC) [5–9]. The expression for the potential in the simplest case of the only bound state at the energy $E = E_{\mathrm{BSEC}}$ is as follows:

$$V_{\mathrm{BSEC}}(x) = -2\,\frac{d}{dx}\left\{\frac{c^2 \sin^2(k_b x)}{k_b^2[1 + \frac{c^2}{k_b^2}(\frac{x}{2} - \frac{\sin(2k_b x)}{4k_b})]}\right\}, \qquad k_b = \sqrt{E_{\mathrm{BSEC}}}\,. \tag{1}$$

This potential is *energy independent.* Do not confuse the following: the quantity $E = E_{\mathrm{BSEC}}$ represents, in expression (1), a *parameter.* The wave function normalized to unity at $E = E_{\mathrm{BSEC}} > 0$ has the form

$$\Psi(x, E = E_{\mathrm{BSEC}}) = \frac{c \sin(k_b x)}{k_b[1 + \frac{c^2}{k_b^2}(\frac{x}{2} - \frac{\sin(2k_b x)}{4k_b})]}\,. \tag{2}$$

The potential (1) and the wave function (2) are shown on the right-hand side of Fig. 1 for $x \geq 0$. They all decrease asymptotically $\sim\frac{1}{x}$ as $x \to \infty$. The parameter c is the derivative of the BSEC function (2) at the origin. The wave function knots at $E = E_{\mathrm{BSEC}}$ coincide with even knots of the potential [and with knots of $\sin(k_b x)$]. This statement follows directly from formulas (1) and (2). From a physical point of view, this conformity between the oscillations of the potential and the BSEC wave function is important for BSEC to exist. The wave confinement mechanism for BSEC was explained in detail in [10]. Besides the BSEC (2), there is another linearly independent solution at $E = E_{\mathrm{BSEC}}$ which diverges asymptotically as $x \to \infty$. The solutions at energies $E \neq E_{\mathrm{BSEC}}$ are the scattering states. Their oscillations are not in "resonance" with $V_{\mathrm{BSEC}}(x)$ and so cannot be confined by it.

Let us now construct the potential $V_r(x)$ on the *entire axis* which vanishes on the negative semiaxis and coincides with $V_{\mathrm{BSEC}}(x)$ (1) on the positive semiaxis

$$V_r(x) = \begin{cases} 0, & x < 0 \\ V_{\mathrm{BSEC}}(x), & x \geq 0 \end{cases} \tag{3}$$

(see Fig. 1).

For this potential, the property needed to retain the BSEC is naturally lost (the waves freely propagate along the negative semiaxis). Instead, the potential

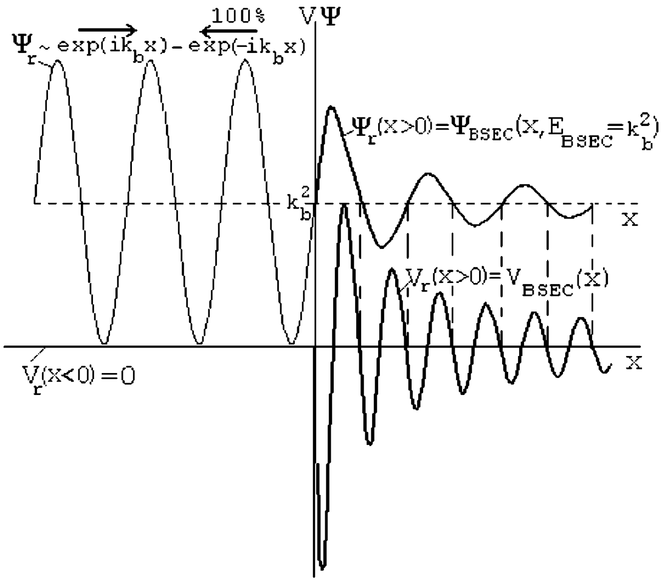

FIG. 1. The scattering wave function corresponding to total reflection at $E = E_{\mathrm{BSEC}} = k_b^2 = 1$; $c = 1$ (solid line) by the potential which is zero on the negative semiaxis and equal to the BSEC potential on the positive semiaxis. The Ψ axis is shifted up so that its origin coincides with the value $k_b^2 = E_{\mathrm{BSEC}} = 1$ of the energy axis. Note the coincidence of BSEC knots with the even knots of the BSEC potential. This exact correlation is important for the wave suppressing for $x > 0$ at the energy value $E = E_{\mathrm{BSEC}} = 1$.

$V_r(x)$ acquires another remarkable property: it *totally* reflects the waves incident from the left at the energy E_{BSEC}. It is because the only physically acceptable solution $\Psi_r(x, E_{\mathrm{BSEC}})$ coincides on the right semiaxis with the $\Psi(x, E = E_{\mathrm{BSEC}})$ vanishing asymptotically as $x \to \infty$. On the negative semiaxis, the solution $\Psi_r(x, E_{\mathrm{BSEC}})$ is a combination of free waves, a concrete form of which is determined from the condition of a smooth matching with solution (2) at the point $x = 0$. We have $\Psi_r(x, E_{\mathrm{BSEC}}) = c \sin(k_b x)/k_b = c[\exp(ik_b x) - \exp(-ik_b x)]/2ik_b, x < 0$, i.e., incident from the left, and outgoing waves in the solution $\Psi_r(x, E_{\mathrm{BSEC}})$ have *equal* amplitudes. As a result, we obtain the unit reflection coefficient modulus $|R(E_{\mathrm{BSEC}})| = 1$. One can see in Fig. 1 that the total reflection energy E_{BSEC} can be above all the barriers of the potential.

At energies $E \neq E_{\mathrm{BSEC}}$, the potential (1) does not keep the waves $\Psi(x, k)$, $k = \sqrt{E}$, and $0 \leq x < \infty$ from propagating to the right. In other words, we have scattering states on the semiaxis outside the energy point E_{BSEC}. For these states, the conformity between the potential and wave oscillations, being necessary for retaining the bound state, is violated. The energy dependence of the reflection coefficient modulus $|R(E)|$ has a resonance character, as is shown in Fig. 2. Greater values of c correspond to wider resonances and vice versa. So one can control the resonance reflection peak width at $E = E_{\mathrm{BSEC}}$ by varying the parameter c. We can also change the number of resonance points and their positions. For this purpose, we must take the potentials with several BSEC states on the positive semiaxis. Such potentials are readily generated by using the inverse technique [7]. The potential (1) considered above is a particular case of the corresponding exactly solvable inverse problem models.

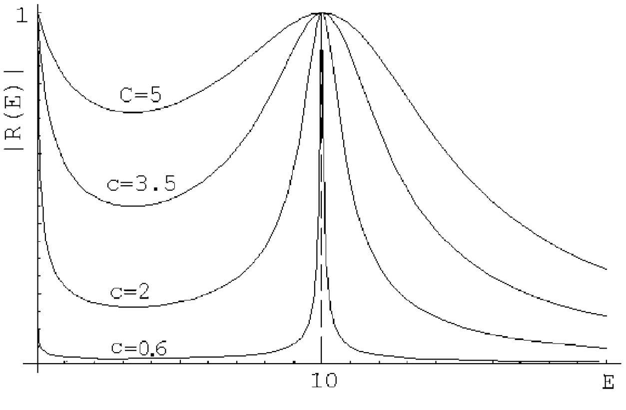

FIG. 2. The modulus of the reflection coefficient $|R(E)|$ for potentials having BSEC on the semiaxis $0 \leq x < \infty$ at energy $E_{\mathrm{BSEC}} = 10$ with resonance reflection at this point for waves on the whole axis. The increase in the BSEC parameter c leads to a greater width of the reflection resonance on the whole axis. With $c \to 0$, this width tends to zero. The dashed line corresponds to the limiting peak in $|R(E)|$.

The manifold of resonance reflection potentials is not exhausted by the interactions considered above. In the case of the whole axis we are not restricted by the condition that $\Psi_r(x = 0) = 0$. The inverse problem formalism allows one to construct the potentials with quadratically integrable states in the continuum with an arbitrary logarithmic derivative at $x = 0$. If continued to the whole axis as in Eq. (3), these potentials are also absolutely reflective at chosen points of the spectrum. And it is not significant that the corresponding states on the right semiaxis have no physical meaning in themselves. The mechanism of suppression of waves on the right semiaxis is the same as for the BSEC.

It should also be added that just the tails of the BSEC potentials beginning arbitrarily far (the intervals $a \leq x < \infty$ for any a) cause, in fact, the wave confinement. Thus, one can add almost any potential from the left (e.g., the δ-function barrier) without moving the position of the reflection resonance, but its shape can be somewhat distorted.

Of course, the periodic potentials on the semiaxis also have the property of total reflection. This occurs at the energy values belonging to the forbidden spectral zones of the same periodic potential continued to the whole axis. Unlike our case, periodic potentials are not quadratically integrable as potentials with BSEC and have the whole bands of total reflection. By the way, perturbation of the periodic potential on the semiaxis can create a BSEC in the allowed spectral zone [7]. For the waves at E_{BSEC} on the whole axis this perturbation will also be totally reflective (we have verified this on a Dirac comb model).

One should not confuse the phenomenon with the total reflection of the waves incident at some angle upon a surface dividing different optical media. In our case, the total reflection is even for incident waves perpendicular to this surface.

The resonance reflection also takes place for a multichannel system of coupled Schrödinger equations for partial channel components Ψ_α of the total wave function [11]:

$$-\Psi''_\alpha(x) + \sum_\beta V_{\alpha\beta}(x)\Psi_\beta(x) = E_\alpha \Psi_\alpha(x), \qquad E_\alpha = E - \epsilon_\alpha . \tag{4}$$

Here the interaction matrix elements $V_{\alpha\beta}$ couple the channels (corresponding to individual equations), ϵ_α represents the threshold energies above which αth channels become open. Such systems often occur in the description of multiparticle quantum systems (nuclear, molecular, etc.). There are more solutions in the multichannel case and these enrich the possibility to find new effects. The direct (matrix) generalization of the one-channel theory is possible. In [9,11], the corresponding analogs of BSEC states on the semiaxis were presented. It was shown that for M coupled equations there could "paradoxically" coexist, for instance, $m \leq M$ scattering solutions with M-m BSEC solutions at the same energy value $E_{\rm BSEC}$. It is clear that by applying these solutions and using an analogous scheme as above we obtain M-m linearly independent scattering solutions on the whole axis with complete reflection at the same energy $E_{\rm BSEC}$. It may seem paradoxical that this can be used for the construction of an exactly solvable model with the simultaneous coexistence of total reflection and absolute transparency at the same energy for the same interaction but for different combinations of incident waves in different channels. A trivial example allows one to understand this: consider, as a two-channel system, a pair of uncoupled one-channel systems, one of which has total reflection and the other has absolute transparency at the same energy. Then, a channel coupling between two uncoupled channels can be switched on without disturbing the features of their continuum spectral branches. This can be done with the creation of a bound state with nonzero partial channel components of the spectral weight vector [7,8,11].

We have revealed that multichannel systems also permit another mechanism of the total resonance reflection. For example, in a two-channel system, incident waves in the first open ($E_1 > 0$) channel partially transit to the second closed ($E_2 < 0$) channel through the interchannel coupling V_{12}. In the second channel these waves are confined in space, i.e., unable to go to infinity [$\Psi_2(x)$ decreases as $\exp(\mp\sqrt{|E_2|}\,x)$ when $x \to \pm\infty$] . The waves returning to the first channel as a result of channel coupling V_{21} propagate in both directions. The decaying waves going in the forward direction can undergo destructive interference with the transmitted waves in the first channel (these waves can have the same amplitudes but opposite phases). This complete wave cancellation which can occur at some energy point implies a total reflection. Corresponding resonances are shown in Fig. 3 for the model with the δ-interaction matrix $V_{\alpha\beta}\delta(x)$. One can control the position E_r of the reflection resonance on the energy scale by choosing the V_{22} value.

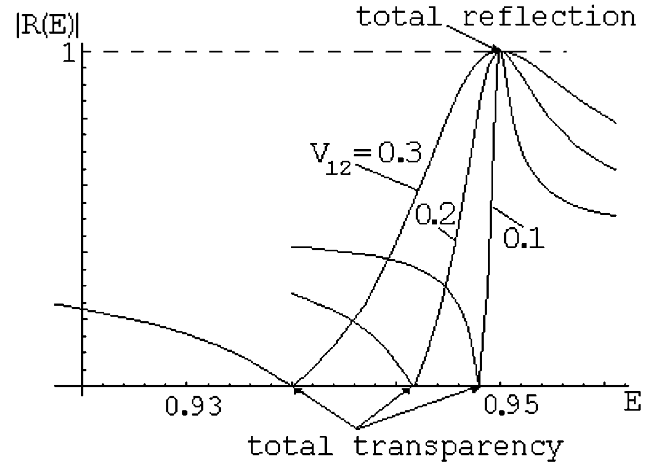

FIG. 3. The energy dependence of the absolute value of the reflection coefficient $|R(E)|$ for waves in the first channel when the second one is closed (its threshold is $\varepsilon_2 = 1$). Parameters of the interaction matrix $V_{\alpha\beta}\delta(x)$ are chosen $V_{11} = 1$; $V_{22} = -0.4$; $V_{12} = 0.1$, 0.2, and 0.3.

The smaller the parameter $V_{21} = V_{12}$ the more slowly the waves return (decay) to the first channel, which promotes the wave accumulation in the second closed channel. Thus the variation of the parameter V_{21} changes the resonance widths (decay rate). The decrease in V_{11} makes the resonance broader.

There are also transparency resonances at the energy points where the phases of the waves decaying from the second channel into the first one are opposite to the phases of the reflected waves. Figure 3 shows just the existence of both the total reflection and transparency at different energies for the same quantum system.

The resonance reflection and transparency also occur in the case of periodic structure of the coupled channels when the energy is chosen in a forbidden zone of the second channel. This was verified by using the model with the Dirac comb periodic potential in the second channel and the δ-interaction matrix $V_{\alpha\beta}\delta(x)$ under a special choice of the interaction parameters $V_{\alpha\beta}$.

The effect of one-channel and multichannel resonance reflections can be used for spectral wave filters. For example, the desired effective one-dimensional potentials can be constructed in different thin waveguides (in particular, quantum wires) with a rarefied spectrum of transverse vibrations when the excitations of higher levels can be neglected. The potential for the longitudinal motion can be formed by modulation of the width along the wire. This is because the squeezing of a waveguide lifts the energy levels of the transverse motion and this effectively increases the potential energy of the longitudinal motion. It is also possible to modulate, by using the law (1), the coordinate dependence of the refractive index occurring in an analog of the Schrödinger equation for optic monochromatic waves. In neutron physics, one can construct the desired effective potentials by employing proper materials and applied modulated magnetic fields.

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