Topological Relaxation of Entangled Flux Lattices: Single versus Collective Line Dynamics

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A symbolic language allowing one to solve statistical problems for the systems with non-Abelian braidlike topology in 2 + 1 dimensions is developed. The approach is based on the similarity between a growing braid and a "heap of colored pieces." As an application, the problem of a vortex glass transition in high-T_c superconductors is reexamined on a microscopic level.

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Statistics of ensembles of uncrossable linear objects with topological constraints has a very broad application area ranging from problems of self-diffusion of directed polymer chains in flows and nematiclike textures to dynamical and topological aspects of vortex glasses in high temperature superconductors [1]. In this Letter we propose a microscopic approach to a diffusive dynamics of entangled uncrossable lines of an arbitrary physical nature.

It is well known that the main difficulties in the statistical topology of linear uncrossable objects are due to two facts: (a) the topological constraints are nonlocal and (b) different entanglements do not commute. The difficulty (a) can be resolved by introducing (Abelian) gausslike topological invariant properly counting windings of one chain around the other, while the circumstance (b) now creates a major problem in the constructive approach to topological theories beyond the Abelian approximation.

We develop a symbolic language which would permit us to construct the objects with a braidlike topology in 2 + 1 dimension and to solve the simplest statistical problems with the noncommutative nature of topological constraints properly taken into account. The results are applied to a reexamination of the problem of a vortex glass transition in high-T_c superconductors [2]. Recall briefly that in CuO_2-based high-T_c superconductors in fields less than H_{c2} there exists a region where the Abrikosov flux lattice is molten, but the sample of the supereconductor demonstrates the absence of the conductivity. This effect is explained by the highly entangled state of flux lines due to their topological constraints [2].

The most attention in our investigation is paid to a quantitative estimation of a characteristic time of self-disentanglement of a particular "test" chain in a bunch of braided directed chains. We distinguish between two situations: (i) all lines in a bunch are parallel, forming a lattice, except one test line randomly entangled with the others, and (ii) no chain is fixed in a bunch of braided lines and any chain winds randomly like a test one. We compute the characteristic disentanglement times $\tau_{\rm si}$ and $\tau_{\rm co}$ in cases (i) and (ii) and demonstrate the *absence of the qualitative difference* between $\tau_{\rm si}$ and $\tau_{\rm co}$. This contradicts in detail with the statement of [2] on the *qualitative difference* between $\tau_{\rm si}$ and $\tau_{\rm co}$ obtained in the framework of a scaling analysis. However, our result does not destroy the physical conclusions of Ref. [2] about the possibility of topological glass transition in the entangled flux state in high-T_c superconductors. According to the above-mentioned cases (i) and (ii) we define, respectively, models I and II.

Model I is as follows: Take a square lattice in the (xy) plane with a spacing c and put the uncrossable obstacles in all vertices of this lattice. Consider a symmetric random walk with the step length c on a dual lattice shifted by $\frac{c}{2}$ in both x and y directions. Let $P_{\rm si}(N)$ be the probability of the fact that after N steps on the dual lattice the random path is closed and disentangled with respect to the obstacles. It is clear that in the (2 + 1)-dimensional [(2 + 1)D] "space-time" $\mathbb{Z}^2(x,y) \times \mathbb{Z}^+(t)$ this model describes the statistics of "world lines" (time-ordered paths) of a single particle jumping on a square lattice in (xy) projection, making each time a step toward the t axis and topologically interacting with the square lattice of infinitely long straight lines. The (xy) section of this model is shown in Fig. 1.

It is known [3] that any topological state of a given path in the lattice of obstacles with coordinational number z is in one-to-one correspondence with some vertex of z-branching Cayley tree. In particular, the N-step trajectory is not entangled with respect to the lattice of obstacles if, and only if, the image of this trajectory on a Cayley tree is closed (i.e., the path on a Cayley tree starts

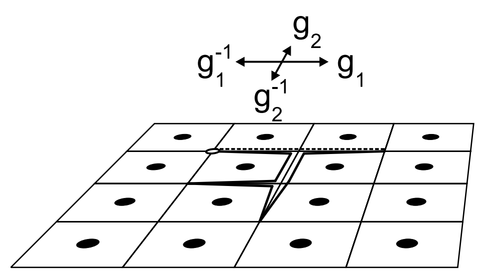

FIG. 1. The (xy) projection of a path (solid line) entangled with a square lattice of obstacles (black points). The primitive path is shown by a dashed line.

at the origin and returns again to the origin of the tree after N steps).

The vertices of the Cayley tree can be naturally parametrized by the words constructed from elementary units—the letters (or "generators") of the group with specific commutation relations. The fact that the Cayley tree has no loops means that there are no commutation relations between letters, and the group associated with this Cayley tree is "free." For example, the 4-branching Cayley tree corresponding to the square lattice of obstacles shown in Fig. 1 is based on an "alphabet"—a set of two generators and their inverses $S = \{g_1, g_2, g_1^{-1}, g_2^{-1}\}$, where $(g_1^{\pm 1}, g_2^{\pm 1})$ do not commute and $g_1 g_1^{-1} = g_2 g_2^{-1} = 1$.

The uniform random walk in the lattice of obstacles is mapped to the random word constructed by writing letters one after another taken from the set S with uniform probability $\frac{1}{4}$. For example, the random walk in Fig. 1 corresponds to the random word W:

$$W = g_1 g_2^{-1} g_1^{-1} g_1 g_2^{-1} g_2 g_2 g_1$$
$$= g_1[\not{g}_2^{-1}(\not{g}_1^{-1}\not{g}_1)(\not{g}_2^{-1}\not{g}_2)\not{g}_2]g_2 = g_1 g_2 .$$

It is easy to show that the length of the shortest (i.e., *irreducible* or *primitive*) word coincides with the shortest (i.e., *geodesic*) distance along the 4-branching Cayley graph of the group Γ_4. In Fig. 1 the primitive path is shown by the dashed line.

In general the average "degree of entanglement" of an N-step path in the lattice of obstacles with coordinational number z for $N \gg 1$ is characterized by the average (geodesic) distance from the root of the Cayley tree with z branches [3] $\langle L_{\rm si}(N)\rangle = \frac{z-2}{z}N$. Hence, the normalized "complexity" $\langle l_{\rm si}\rangle$ of a typical topological state of a path can be defined as

$$\langle l_{\rm si}\rangle \equiv \lim_{N\to\infty}\frac{\langle L_{\rm si}(N)\rangle}{N} = \frac{z-2}{z}. \quad (1)$$

The probability $P_{\rm si}(N)$ to find an N-step random walk in a trivial topological state in the lattice of obstacles coincides with the probability to find the randomly generated N-letter word completely reducible. For diffusion on a z-branching Cayley tree, $P_{\rm si}(N)$ determines the probability for an N-step symmetric random walk to return to the origin. This probability has been computed many times—see, for example [4],

$$P_{\rm si}(N) = \frac{2\sqrt{2}\,p}{\sqrt{\pi}(1-4pq)}\frac{\alpha^N}{N^{3/2}}, \quad (2)$$

where $q = \frac{1}{z}$ and $p \equiv 1 - q = \frac{z-1}{z}$ are the probabilities of steps "forward" to and "backward" from the origin of the Cayley tree; $\alpha = 2\sqrt{pq}$. Λ is defined as a span of the N-step random walk in (xy) projection. In physical terms of the original "vortex problem," $\sqrt{\Lambda^2}$ is the average size of thermal fluctuations of the vortex line. Hence, we can set $N = \frac{\Lambda^2}{c^2}$ and estimate the typical time of disentanglement $\tau_{\rm si} = \frac{1}{P_{\rm si}(N)}$ of a single vortex line in an ensemble of immobile uncrossable lines for $z = 4$ as follows:

$$\tau_{\rm si} \sim \left(\frac{\Lambda^2}{c^2}\right)^{3/2}\left(\frac{2}{\sqrt{3}}\right)^{\Lambda^2/c^2}. \quad (3)$$

In contrast to model I, *model II* describes collective dynamics of the world lines and ultimately leads to the consideration of the (2 + 1)-dimensional ("surface") *braid group* $B_{n+1}^{\rm 2D}$. The group $B_{n+1}^{\rm 2D}$ has $2n^2 + 2n$ generators $\sigma_{ij}^{(x)}, \sigma_{ij}^{(y)}$ and their inverses with standard "braiding" relations [5] ($\{i,j\} \in [1, n+1]$). The geometric representation of generators of $B_{n+1}^{\rm 2D}$ is shown in Fig. 2a. Graphically the braid is represented by a set of strings, going upwards in accordance with the growth of a braid length—see Fig. 3a. An element of the braid group $B_{n+1}^{\rm 2D}$ is set by a word in the alphabet $\{\sigma_{11}^{(x)}, \sigma_{11}^{(y)}, \ldots\}$. By the *length N* of a braid, we call the length of a word in a given record of the braid and, by the irreducible length (or *primitive length*), we call the shortest length of a word in which the braid can be written. The irreducible length can also be viewed as a distance from the unity on the graph of the group $B_{n+1}^{\rm 2D}$—in the same manner as for a Cayley tree.

We now define a symmetric random walk on a set of generators ("letters") $\{\sigma_{11}^{(x)}, \sigma_{11}^{(y)}, \ldots\}$ with the transition probability $\frac{1}{2n^2+2n}$. Namely, we raise recursively an N-letter random word (i.e., random braid) W, adding step-by-step the letters (say, from the right-hand side) to a growing word. The probability that the N-letter word is completely contractible (i.e., has a zero's primitive length) defines the probability to have a topologically trivial braid of the record length N.

Our main tool in the investigation of the braid group is the so-called *locally free group* [6]. The (2 + 1)D (surface) locally free group $\mathcal{LF}_n^{\rm 2D}$ is obtained from the braid group $B_{n+1}^{\rm 2D}$ by omitting the braiding relations. Thus the group $\mathcal{LF}_n^{\rm 2D}$ has $2n^2 + 2n$ generators, $f_{ij}^{(x)}$, $f_{ij}^{(y)}$, and their inverses, ($\{i,j\} \in [1, n+1]$). The geometric interpretation of generators is given in Fig. 2a. The commutation relations are set by the rules: nearest-neighbor generators (letters) do not commute, while distant generators do. Schematically these rules are depicted in Fig. 4.

There is a one-to-one correspondence between words in the locally free group and *colored heaps* whose elements are either "white," $f_{ij}^{(x,y)}$, or "black," $(f_{ij}^{(x,y)})^{-1}$. Any word

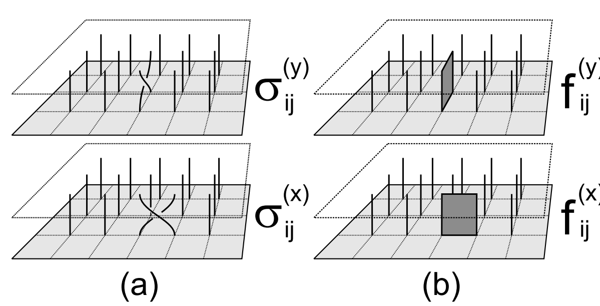

FIG. 2. The generators of the surface groups $B_n^{\rm 2D}$ (a) and $\mathcal{LF}_{n+1}^{\rm 2D}$ (b).

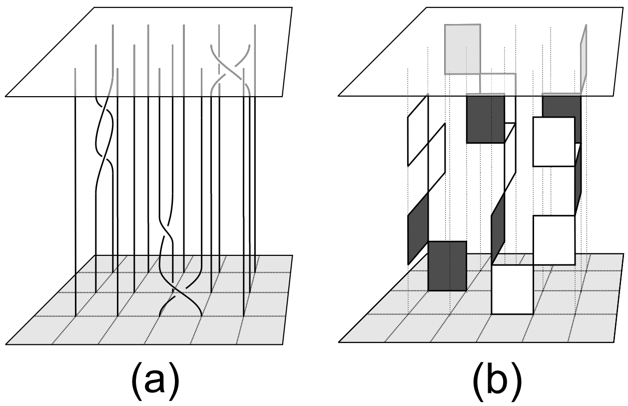

FIG. 3. (a) The (2 + 1)D braid. (b) The (2 + 1)D colored heap.

written in terms of letters-generators of the group $\mathcal{L}\mathcal{F}_n^{2D}$ represents a configuration of a colored heap (see Fig. 3b) in a box at the base of $n \times n$ cells. And, vice versa, any heap uniquely defines some sequence of letters, i.e., a word in the group $\mathcal{L}\mathcal{F}_n^{2D}$. The configuration of a heap with black elements following immediately after white ones in the same column is forbidden.

For the uniform Markov dynamics on the set of braid group generators we can compute the average primitive (i.e., irreducible) length $\langle L_{\rm co}(N)\rangle$ of the N-letter word, characterizing the degree of entanglement of threads in a braid [compare to the definition of $\langle L_{\rm si}(N)\rangle$]. Let us show how the concept of the locally free group can help in estimating $\langle L_{\rm co}(N)\rangle$.

The group $\mathcal{L}\mathcal{F}_n^{2D}$ has less relations than B_{n+1}^{2D}. Hence, the number of distinct words of primitive length $L_{\rm co}(N \mid B_{n+1}^{2D})$ in the braid group *is bounded from above* by the number of distinct words of the same primitive length $L_{\rm co}(N \mid \mathcal{L}\mathcal{F}_n^{2D})$ in the locally free group. Thus, some words being irreducible in the group $\mathcal{L}\mathcal{F}_n^{2D}$ can be reduced by applying extra braiding relations from the group B_{n+1}^{2D}, and the following inequality holds:

$$L_{\rm co}(N \mid B_{n+1}^{2D}) \le L_{\rm co}(N \mid \mathcal{L}\mathcal{F}_n^{2D}). \qquad (4)$$

At the same time, by construction (compare Figs. 2a and 2b) $f_{i,j}^{(x,y)} = (\sigma_{i,j}^{(x,y)})^2$. Thus, the number of distinct words of length $2L_{\rm co}$ in the braid group *is bounded from below* by the number of distinct words of the length $L_{\rm co}$ in the locally free group [6], which results in the inequality

$$\langle L_{\rm co}(N \mid \mathcal{L}\mathcal{F}_n^{2D})\rangle \le 2\langle L_{\rm co}(N \mid B_{n+1}^{2D})\rangle. \qquad (5)$$

Equations (4) and (5) allow us to get the bilateral estimation for the average primitive length $\langle L_{\rm co}(N \mid B_{n+1}^{2D})\rangle$ of the N-step random walk on the surface braid group B_{n+1}^{2D} and hence to measure the average degree of entanglement in a bunch of randomly wreathed directed lines.

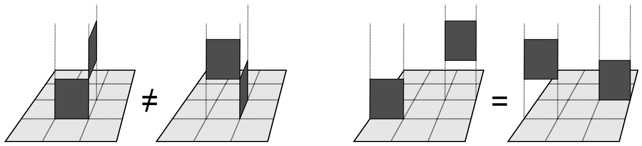

FIG. 4. Commutation relations among generators of a (2 + 1)-dimensional locally free group.

The computation of $\langle L_{\rm co}(N \mid \mathcal{L}\mathcal{F}_n^{2D})\rangle$ involves the concept of the *roof of the heap* [7]. In physical terms a roof consists of a set of "topmost" elements which can be removed from the heap without disturbing the rest. The projection of a roof onto the (xy) plane for some particular configuration of the topmost elements is shown in Fig. 5. Let us stress that local heights (measured from the bottom of the box) of different roof blocks might be different.

The process of growth of a heap (i.e., the random walk on the group $\mathcal{L}\mathcal{F}_n^{2D}$) consists of randomly adding step-by-step new black or white blocks to the roof [7,8]. Hence the dynamics of a heap is controlled by the dynamics of a roof. For a particular configuration of a roof we define the "size" of a roof T (i.e., the number of marked segments in Fig. 5) and the number of empty segments n_i having i marked neighbors (apparently, $n_i = 0\ \forall\, i \ge 3$). In Fig. 5, one has $n = 5$, $n_0 = 6$, $n_1 = 25$, $n_2 = 17$; $T = 12$. The values T, n_i obey the conservation conditions:

$$\begin{cases} n_0 + n_1 + n_2 + T = 2n^2 + 2n \\ 6T - 8(n+1) \le n_1 + 2n_2 \le 6T \end{cases}. \qquad (6)$$

For any configuration the dynamics of a roof's size is described by local increments ΔT:

$$\begin{cases} \Delta T = +1 & \text{with probability } \frac{n_0}{2n^2+2n} \\ \Delta T = 0 & \text{with probability } \frac{n_1+T}{2n^2+2n} \\ \Delta T = -1 & \text{with probability } \frac{n_2}{2n^2+2n} \end{cases}.$$

Taking Eq. (6) into account, we can write

$$\Delta T = \frac{n_0 - n_2}{2n^2 + 2n} = \frac{2n^2 - (n_1 + 2n_2)}{2n^2 + 2n}.$$

In a stationary case, one has $\langle \Delta T\rangle = 0$, which permits one to get, in the limit $n \gg 1$, the asymptotics of the average roof size $\langle T\rangle$:

$$\langle T\rangle = \frac{2n^2}{7}. \qquad (7)$$

Dynamics of the roof determines the dynamics of the whole heap. Return to the random walk on $\mathcal{L}\mathcal{F}_n^{2D}$ and compute the conditional change of the primitive length $L_{\rm co}(N \mid \mathcal{L}\mathcal{F}_n^{2D})$ for one step of the random walk:

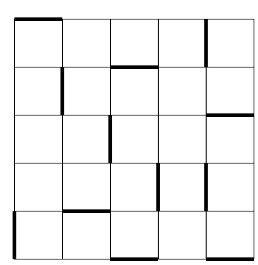

FIG. 5. A particular configuration of a roof of (2 + 1)D heap is shown by marked segments.

$$\begin{cases} \Delta L_{\rm co}(N \,|\, \mathcal{L}\,\mathcal{F}_n^{\rm 2D}) = +1 & \text{with probability } 1 - \frac{T}{2(2n^2+2n)} \\ \Delta L_{\rm co}(N \,|\, \mathcal{L}\,\mathcal{F}_n^{\rm 2D}) = -1 & \text{with probability } \frac{T}{2(2n^2+2n)} \end{cases} . \quad (8)$$

Using (7), one obtains, in a steady state for $N \gg n \gg 1$,

$$\langle L_{\rm co}(N \,|\, \mathcal{L}\,\mathcal{F}_n^{\rm 2D})\rangle = N\langle \Delta L_{\rm co}(N \,|\, \mathcal{L}\,\mathcal{F}_n^{\rm 2D})\rangle = N\left[1 - \frac{\langle T\rangle}{2n^2 + 2n}\right] = \frac{6}{7}N . \quad (9)$$

Thus, according to (4) and (5), one arrives at the bilateral estimation of the average length of the primitive word for the N-step random walk on the surface braid group

$$\frac{3}{7} \le \lim_{\substack{N\to\infty \\ n\to\infty}} \frac{\langle L_{\rm co}(N \,|\, B_n^{\rm 2D})\rangle}{N} \le \frac{6}{7} . \quad (10)$$

The quantity $\langle l_{\rm co}\rangle = \lim_{\substack{N\to\infty \\ n\to\infty}} \frac{\langle L_{\rm co}(N \,|\, B_n^{\rm 2D})\rangle}{N}$ characterizes the complexity of the entangled state of threads. By comparing (10) to (1), one concludes that the average topological state of a braid $\langle L_{\rm co}(N \,|\, B_n^{\rm 2D})\rangle$ obtained in the course of collective motion of all lines has the same asymptotics in N as the one from a single line motion and interpolates between entangled states in effective lattices of obstacles with coordinational numbers $z_{\rm eff}$, where

$$\tfrac{7}{2} \le z_{\rm eff} \le 14 . \quad (11)$$

Equations (3), (10), and (11) allow us to estimate *from above* the characteristic time of disentanglement in a bunch of vortex lines, considered as a braid of directed random walks. Recall that the random growth of a braid is interpreted as an N-step uniform random walk on a surface braid group $B_{n+1}^{\rm 2D}$. The topological state of a braid is uniquely characterized by a primitive (irreducible) word w_N in terms of generators of $B_{n+1}^{\rm 2D}$. The disentangled state of two neighboring trajectories means (for the group $B_{n+1}^{\rm 2D}$) that the primitive word w_N does not contain the corresponding neighboring generators, say $\sigma_{i,j}^{(x)}$, $\sigma_{i,j}^{(y)}$, $(\sigma_{i,j}^{(x)})^{-1}$, and $(\sigma_{i,j}^{(x)})^{-1}$ for some $(i,j) \in 1, n+1$.

We compute the typical time $\tau_{\rm co}$ of disentanglement of two neighboring threads in a braid, appealing to the geometrical image instead of playing with rigorous algebraic estimates. It seems to be evident that the time of disentanglement of any part of a braid can be roughly estimated from above by the time of complete disentanglement of all threads in a bunch. In terms of a random walk on a braid group $B_{n+1}^{\rm 2D}$, the simultaneous disentanglement of all lines means that the primitive word w_N has zero length. By using the established relation between braids, heaps, and effective lattices of obstacles reflected in (11), one can claim the following upper estimate for $\tau_{\rm co}$ computed at $z_{\rm eff} = 14$:

$$\tau_{\rm co}^{\rm up} \sim \left(\frac{\Lambda^2}{c^2}\right)^{3/2}\left(\frac{7}{\sqrt{13}}\right)^{\Lambda^2/c^2} . \quad (12)$$

Comparing (2) and (12), one has $\lim_{\frac{\Lambda}{c}\to\infty} \frac{c^2}{\Lambda^2} \ln\frac{\tau_{\rm co}}{\tau_{\rm si}} \approx 0.5$.

In conclusion let us emphasize that the random braiding (model II) can be analyzed within the framework of a symbolic dynamics on the locally free group describing the growth of a heap of colored pieces. The probability of having a disentangled state of two vortex lines can be estimated from above by the probability of having no pieces (elements) in a given column of a heap. This model seems to be a natural discretization of a standard ballistic deposition process of a Kardar-Parisi-Zhang–type (KPZ-type) [9]. In the context of a microscopic approach developed above, one can easily check whether the scaling conjecture of [2] is correct. Considering the collective dynamics of vortex lines, Obukhov and Rubinstein supposed the existence of some macroscopic domains where all neighboring vortices are mutually disentangled (Fig. 3 in [2]). In our terms, the existence of such disentangled configurations of vortex lines leads to strong fluctuations of the roof's width $\langle \delta h\rangle$. However, it is known for KPZ-type models that fluctuations of the roof are bounded: $\lim_{\langle h\rangle\to\infty} \frac{\langle \delta h\rangle}{\langle h\rangle} = 0$, where $\langle h\rangle$ is the average height of a heap, which hints at a possible source of inconsistency in the scaling arguments of [2].

Moreover, model II can be easily modified to take into account the possibility of the vortex line breaking. The rupture of vortex lines relaxing the entangled state can be modeled by random "recoloring" of elements: white → black and black → white in an already created heap, which leads to additional cancellation of the heap's elements. By defining the probability $r = e^{-E/T}$ of a random recoloring of an element of a heap (E is associated with an energy of vortex line rupture), we modify the probability in the last line of (8): $\frac{T}{2(2n^2+2n)} \to \frac{T}{2(2n^2+2n)} + r$ which leads to $\langle L_{\rm co}(N \,|\, \mathcal{L}\,\mathcal{F}_n^{\rm 2D})\rangle = N(\frac{6}{7} - r)$ [compare to (9)]. As one can see, for $E < E_{\rm cr} = T\ln\frac{7}{6}$, one has $\langle L_{\rm co}(N \,|\, \mathcal{L}\,\mathcal{F}_n^{\rm 2D})\rangle = o(N)$ and hence the heavily entangled state is completely relaxed.

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