

## Quantum Critical Points with the Coulomb Interaction and the Dynamical Exponent: When and Why $z = 1$

Igor F. Herbut

*Department of Physics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6*

(Received 1 June 2001; published 10 September 2001)

A general scenario that leads to Coulomb quantum criticality with the dynamical critical exponent  $z = 1$  is proposed. I point out that the long-range Coulomb interaction and quenched disorder have competing effects on  $z$ , and that balance between the two may lead to charged quantum critical points at which  $z = 1$  exactly. This is illustrated with the calculation for the Josephson junction array Hamiltonian in dimensions  $D = 3 - \epsilon$ . Precisely in  $D = 3$ , however, the above simple result breaks down, and  $z > 1$ . Relation to other studies is discussed.

DOI: 10.1103/PhysRevLett.87.137004

PACS numbers: 74.40.+k, 05.30.-d, 71.30.+h, 73.50.-h

The crucial difference between the quantum ( $T = 0$ ) and the more familiar finite temperature phase transitions is that while dynamics is irrelevant for the latter, it is essential for the former [1]. The link between statics and dynamics at a continuous quantum phase transition is usually parametrized with the value of *dynamical* critical exponent  $z$  that describes relative scaling of the time and the length scales in the problem [2,3]. Together with the correlation length exponent  $\nu$ ,  $z$  enters the low-temperature scaling of all physical observables, since in the vicinity of a quantum critical point temperature scales as  $T \sim |\delta|^{z\nu}$ , where  $\delta$  is the  $T = 0$  tuning parameter. The value of  $z$  for a given quantum critical point is therefore of great interest, and has often been used to distinguish one universality class from another. In their seminal paper on universal conductivity in two dimensions, Fisher, Grinstein, and Girvin [4] also proposed that when the long-range Coulomb interaction is present, at the criticality energy should scale as the inverse of length, so that  $z = 1$  should result. This well-known conjecture has since been used in interpreting some of the most intriguing experiments in modern condensed matter physics, ranging from superconductor-insulator transitions in low-dimensional systems [5], via metal-insulator transitions in Si-metal-oxide-semiconductor field-effect transistors [6], to the universality of the underdoped high- $T_c$  cuprates [7]. It has also been utilized in the Monte Carlo simulations where knowing  $z$  in advance greatly simplifies the inevitable finite-size scaling analysis [8].

The purpose of this Letter is to provide the theoretical justification for this widely used relation and point to its limitations. I show that in certain dimensions the Coulomb coupling constant (i.e., charge) is protected from renormalization, and consequently its flow under scaling transformation is directly proportional to its canonical dimension, which is just  $z - 1$  [4] [see Eq. (2)]. This implies that *if* a charged critical point exists in the theory, its location is actually determined by the solution of the equation  $z = 1$ , which then also determines the value of the dynamical critical exponent exactly. I argue that this situation arises when there is an additional coupling in the theory with the competing effect on the dynamical

exponent, and which can balance the effect of Coulomb interaction. Such a coupling is shown to be provided by quenched disorder, and a concrete realization of the above scenario is worked out on the example of a disordered Josephson junction array Hamiltonian in  $D = 3 - \epsilon$  dimensions. Finally, the simple relation  $z = 1$  is found to break down at special dimensions at which the above renormalization group (RG) protectorate on charge is lifted. Relation to other recent theoretical studies of Coulomb criticality is discussed.

To be specific, I will focus on the theory originally considered in [9] which describes an array of coupled Josephson junctions, but it will transpire that the underlying physics is more general. Building on an earlier work by Ma [10], Fisher and Grinstein [9] have shown that Coulomb interaction can be represented by a minimal coupling to the soft *scalar* gauge field. In the long wavelength limit the critical field theory that describes the system of bosons interacting via Coulomb interaction at  $T = 0$  and in  $D$  dimensions takes the form [9]

$$S = \int d^D \vec{r} d\tau \left\{ |(\partial_\tau - iA_0)\Psi|^2 + |\nabla\Psi|^2 + [V(\vec{r}, \tau) + m^2]|\Psi|^2 + \lambda|\Psi|^4 + \frac{1}{2e} A_0 |\nabla|^{D-1} A_0 \right\}, \quad (1)$$

where by  $|\nabla|^{D-1}$  it is meant  $|\vec{k}|^{D-1}$  in the Fourier space.  $\Psi(\vec{r}, \tau)$  is the complex superfluid order parameter, and  $A_0(\vec{r}, \tau)$  is the scalar gauge field, which when integrated out introduces the Coulomb interaction into the remaining action for  $\Psi$ . I have also included a random potential  $V(\vec{r}, \tau)$ , but more on this in a moment. More generally, the gauge-field propagator in the momentum space is  $\{e^2[V_c(k) - 1]\}$  [9], which for the Coulomb interaction  $V_c(\vec{r}) = 1/r$  then yields the last term in (1) at small momenta. The presence of a neutralizing background is included by omitting the  $k = 0$  components of the gauge field [10].

The above action without randomness [ $V(\vec{x}, \tau) \equiv 0$ ] was first studied by Fisher and Grinstein [9], and recently

revisited by Ye [11]. In  $D = 3$  both the charge ( $e^2$ ) and the quartic interaction ( $\lambda$ ) are marginally irrelevant, and the Coulomb interaction causes the flow to run away to negative  $\lambda$ , which may be interpreted as a sign of a discontinuous transition. In  $D = 3 - \epsilon$  the result is more interesting: charge is irrelevant at the XY critical point, but if too large may still lead to a runaway flow. Irrelevance of the charge in the theory (1) arises in a somewhat nontrivial way, and it will prove instructive to understand this in some detail. In  $D < 3$  the inverse gauge-field propagator in (1) is *nonanalytic* in  $k$  and therefore  $e^2$  cannot get renormalized by the integration over the high-energy modes. This RG protectorate is reminiscent of the situation in the  $(2 + 1)$ -dimensional electrodynamics with the Chern Simons term [12], where the statistical angle is exactly marginal for a similar reason. Here this means that the  $\beta$  function for the charge is determined exclusively by its canonical dimension. Gauge invariance and the accompanying Ward identity imply that  $A_0 \sim \tau^{-1}$ , so assuming  $\tau \sim L^z$  yields to

$$\frac{de^2}{d \ln(b)} = e^2(z - 1) \quad (2)$$

in  $D < 3$ , where  $b$  is the standard RG parameter. The exponent  $z$  in the last equation then needs to be determined from the renormalization of the  $\Psi$  propagator. The simplest one-loop calculation [see the first diagram on Fig. 1(a)] in  $D = 3 - \epsilon$  then gives  $z = 1 - e^2/3$ , and thus smaller than one. By Eq. (2) the small charge is then irrelevant. Two features of this result that are likely to

be quite general should be noticed: first,  $z \neq 1$  since the scalar gauge field couples only to the time derivative, and thus discriminates between space and time. Second, the *negative* sign in Eq. (2) comes from the tendency of the gauge field to *soften* the  $\Psi$  propagator. It is analogous to the well-known (and much debated) result that the anomalous dimension in the scalar (Higgs) electrodynamics is negative [13–15]. It could therefore be expected that when higher order terms in  $e^2$  are included  $z < 1$  will remain, and that irrelevancy of the charge in the pure theory (1) is more general than the simple one-loop calculation would suggest.

Equation (2) also implies that if there would exist a critical point in the theory with nonzero charge, in  $D < 3$  at that fixed point  $z = 1$  exactly. This nonperturbative result parallels another exact result  $\eta_A = 1$  in the  $D = 3$  scalar electrodynamics [13,16] where  $\eta_A$  is the anomalous dimension of the vector gauge field. The crucial question is how can such a Coulomb critical point (with  $e^2 \neq 0$ ) arise. A clue is provided already in the preceding discussion: one needs another coupling which will tend to increase  $z$  and balance the effect of Coulomb interaction in Eq. (2). A physically realistic candidate is disorder: in the quantum theory the random potential  $V(\vec{r}, \tau)$  is random in space but static and independent of imaginary time. This anisotropy will in general lead to a nontrivial  $z$ , and it is well known that the effect to the lowest order is always to *increase* it [17]. In other words, while the weak Coulomb interaction is marginally irrelevant at the XY critical point, I expect it to become *relevant* at the disordered critical point where  $z > 1$ . The small charge near the random critical point should grow until it balances disorder in Eq. (2). As a result, a new stable (Coulomb) critical point may arise, at which  $z = 1$  exactly.

Next I demonstrate that the above scenario is indeed born out in the one-loop calculation in the theory (1), in passing reconciling the apparently different results in Refs. [9] and [11]. I then proceed to show how the equality  $z = 1$  is violated in  $D = 3$ , and comment on relations to other works.

To exert some control over the fixed points in the theory I will assume both a small  $\epsilon = 3 - D$  and  $\epsilon_\tau$ , where the latter is the number of dimensions over which disorder is correlated [18]. The physically interesting case corresponds to  $\epsilon_\tau = 1$ , but since I am primarily interested in the point of principle, convergence properties of the double- $\epsilon$  expansion will not be of much concern here [19]. To average over disorder I utilize the standard replica trick, which attaches a replica index  $\alpha = 1, \dots, N$  onto all fields in the action (1) and introduces another interactionlike term in the theory

$$- \frac{W}{2} \sum_{\alpha, \beta=1}^N \int d^{D+1} \vec{R} d^{D+1} \vec{R}' \delta^{D+1-\epsilon_\tau}(\vec{R} - \vec{R}') \times |\Psi_\alpha(\vec{R})|^2 |\Psi_\beta(\vec{R}')|^2, \quad (3)$$

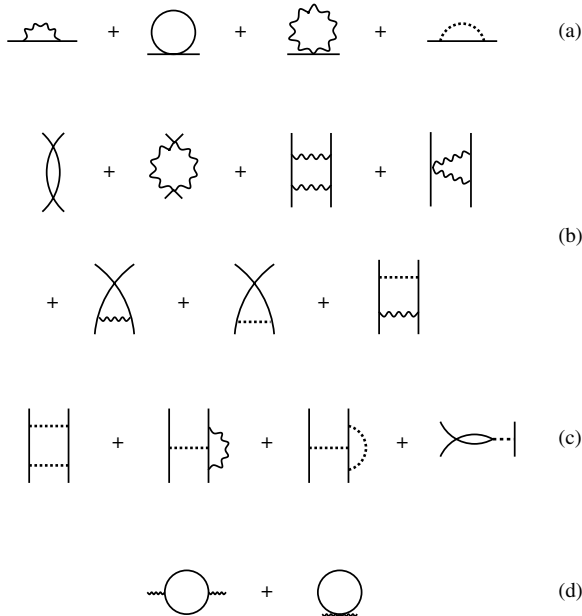


FIG. 1. One-loop contributions to the renormalization of self-energy, quartic interaction, disorder coupling, and polarization. Wavy lines represent the gauge-field propagator, and dashed lines the disorder vertex.

where the limit  $N \rightarrow 0$  is to be taken at the end of the calculation [18]. Here  $\vec{R} = (\vec{r}, \tau)$ , and for  $\epsilon_\tau = 1$  one recovers the quantum problem where disorder is uncorrelated (Gaussian) in space and independent of the imaginary time.

To perform the Wilson-Fisher momentum shell renormalization group one integrates out all the fields with  $\Lambda/b < k < \Lambda$  and  $-\infty < \omega < \infty$ , where  $\Lambda$  is the ultraviolet cutoff. The effect of this procedure is to alter the coefficients in front of  $\omega^2$  ( $Z_\omega$ ),  $k^2$  ( $Z_k$ ),  $m^2$  ( $Z_m$ ),  $\lambda$  ( $Z_\lambda$ ),  $W$  ( $Z_W$ ), and  $A_0^2$ -term ( $Z_A$ ) in the Fourier transformed action (1) [20]. One then rescales the momenta and the frequencies as  $bk \rightarrow k$  and  $b^z \omega \rightarrow \omega$ , and the fields as  $b^{-D} A_0 \rightarrow A_0$  and  $b^{-(2+D+z)/2} Z_k^{1/2} \Psi \rightarrow \Psi$ , to find finally that by defining new coupling constants as  $\lambda(b) = b^{4-D-z} Z_k^{-2} Z_\lambda \lambda$ ,  $W(b) = b^{4-D-z+\epsilon_\tau} Z_k^{-2} Z_W W$ , and  $e^2(b) = b^{z-1} Z_A^{-1} e^2$  the action can be restored into its original form (at the critical point  $m^2 = 0$ ). Computing next the  $Z$  factors diagrammatically to one-loop order (Fig. 1) gives at the criticality

$$\frac{de^2}{d \ln(b)} = e^2(z-1) - \delta_{3,D} \frac{1}{12} e^4, \quad (4)$$

$$\frac{d\lambda}{d \ln(b)} = \left( \epsilon + \frac{1}{2} e^2 + \frac{11}{8} W \right) \lambda - \frac{5}{2} \lambda^2 - \frac{1}{4} e^4, \quad (5)$$

$$\frac{dW}{d \ln(b)} = \left( \epsilon + \epsilon_\tau - 2\lambda + \frac{1}{2} e^2 \right) W + \frac{7}{8} W^2. \quad (6)$$

The exponent  $z$  in Eq. (4) is determined by demanding that  $b^{-2z} Z_\omega = b^{-2} Z_k$ , which gives

$$z = 1 + \frac{1}{8} W - \frac{1}{3} e^2. \quad (7)$$

Note that for  $D < 3$  the flow equation for the charge reduces to Eq. (2). Precisely in  $D = 3$  the inverse propagator for the gauge field becomes analytic,  $\sim k^2$ . This means that in  $D = 3$  the charge becomes renormalized by the polarization diagrams in Fig. 1(d), which contribute the last term in Eq. (4). For  $W = 0$  the preceding  $\beta$  functions reduce to those of Ref. [9] when  $D < 3$ , and coincide with those of [11] right at  $D = 3$ , upon simple redefinitions of couplings. They are also equivalent to those of [19] for  $e^2 = 0$ . It may be interesting to note that many of the individual diagrams on Fig. 1 are ultraviolet divergent, due to the independence of the gauge-field propagator on frequency. All those divergences exactly cancel out in the final result [11]. Finally, the flow of the mass term in (1) yields the correlation length exponent:

$$\nu = \frac{1}{2} + \frac{1}{4} \left( \lambda + \frac{e^2}{6} - \frac{W}{4} \right). \quad (8)$$

Let us turn now to the fixed points of the above equations. Besides the Gaussian and the XY fixed points at  $W = e^2 = 0$ , both unstable with respect to disorder,

there are two disordered fixed points. First, at  $e^2 = 0$ , there is a *neutral* disordered fixed point [18] at  $\lambda_n = 2(4\epsilon + 11\epsilon_\tau)/9$  and  $W_n = 8(\epsilon + 5\epsilon_\tau)/9$ , which is attractive in the  $\lambda - W$  plane. At the neutral fixed point  $z_n = 1 + W/8 > 1$ , so a weak Coulomb interaction is a relevant perturbation. With a small charge turned on, the flow is towards a new, stable, Coulomb critical point. In  $D < 3$  this fixed point is located at

$$\lambda_c = \frac{100}{183} \epsilon \left\{ 1 + \frac{389\epsilon_\tau}{100\epsilon} + \sqrt{\left( 1 + \frac{389\epsilon_\tau}{100\epsilon} \right)^2 + \frac{1647}{5000} \left( 1 + \frac{\epsilon_\tau}{\epsilon} \right)^2} \right\}, \quad (9)$$

$$W_c = \frac{16}{17} (2\lambda_c - \epsilon - \epsilon_\tau), \quad (10)$$

and

$$e_c^2 = \frac{3}{8} W_c, \quad (11)$$

where the last equation ensures that  $z = 1$ . The reader should note that disorder is necessary for the existence of the Coulomb critical point: without it the critical point would turn imaginary, and one would find only the standard runaway flow characteristic of the gauge-field fluctuations in the  $\epsilon$  expansion [21]. The Coulomb critical point therefore may be considered as an example of a disorder *induced* continuous phase transition.

Precisely in  $D = 3$  the last term in Eq. (4) becomes finite. There still exists a stable Coulomb critical point at  $\lambda_c = 3.61\epsilon_\tau$ ,  $W_c = 40(2\lambda_c - \epsilon_\tau)/41$ , and  $e_c^2 = 3W_c/10$ , but with the dynamical critical exponent

$$z = 1 + \frac{e_c^2}{12}, \quad (12)$$

which gives  $z \approx 1.15$ , for example, for  $\epsilon_\tau = 1$ . One finds  $z \neq 1$  in  $D = 3$  as a result of the removal of the RG protectorate on charge. The same violation of the simple relation  $z = 1$  can be expected in other problems in special dimensions. Also, since at the criticality  $V_c(r) \sim 1/r^z$ , one expects that  $z \geq 1$  in general [4], since screening should certainly not make the interaction longer ranged. The result (12) thus implies the Coulomb interaction has been partially screened at the criticality in  $D = 3$ , and now decays faster (but still as a power law) with distance [22].

The result  $z = 1$  was previously also found in the large- $N$  theory of Dirac fermions in  $D = 2$  [23], interacting both via Coulomb interaction and with the Chern Simons field. This theory may be relevant to the quantum Hall state-insulator transition [2,24]. It was found that for a certain range of the statistical angle (Chern Simons coupling) the competition between the Chern Simons and the Coulomb interactions leads to a nontrivial charged fixed point, at which  $z = 1$  exactly, just as in Eq. (2). The physical reason why the two interactions have competing effects on  $z$  only for some values of the Chern Simons coupling

remained somewhat obscure in this work. Nevertheless, the result in [23] bears a formal resemblance to mine.

Finally,  $z = 1$  was also found in the previous work by the author on the quantum critical behavior of dirty bosons with Coulomb interaction in  $D = 1 + \epsilon$  dimensions [25]. There it arises as a consequence of a special symmetry the theory dual to (1) possesses precisely in  $D = 1$ , and can therefore be suspected to be an artifact of the specific RG scheme that was employed. This Letter can thus be understood as complementing the previous work in that it shows that  $z = 1$  is also exact near the  $D = 3$ , and it may therefore be expected to hold in the physical case  $D = 2$ .

It may also be interesting to note that simply setting  $\epsilon = \epsilon_\tau = 1$  in the one-loop Eqs. (7) and (8), to crudely estimate the exponents in  $D = 2$ , besides the exact  $z = 1$  also yields  $\nu = 1.46$  at the Coulomb criticality. Experimentally,  $\nu \approx 1$  [5], and the result for  $\nu$  is less reliable than the one obtained in the expansion around  $D = 1$  [25].

To summarize, it was shown that the competition between the correlated (quantum) disorder and Coulomb interaction may lead to a new charged critical point at which the dynamical critical exponent  $z = 1$  exactly. This simple result breaks down in special dimensions, and an example of the Josephson junction array when this happens in  $D = 3$  was provided. I argued that similar results should be expected whenever there are two couplings in the theory with competing effects on  $z$ .

The author is grateful to Matthew Fisher and Shivaji Sondhi for their useful comments and interest in this work. This research was supported by NSERC of Canada and the Research Corporation.

- 
- [1] John A. Hertz, Phys. Rev. B **14**, 1165 (1976).  
 [2] S. L. Sondhi, S. M. Girvin, J. P. Carini, and D. Shahar, Rev. Mod. Phys. **69**, 315 (1997).  
 [3] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, U.K., 1999).

- [4] M. P. A. Fisher, G. Grinstein, and S. M. Girvin, Phys. Rev. Lett. **64**, 587 (1990).  
 [5] For a review, see A. Goldman and N. Marković, Phys. Today **51**, No. 11, 39 (1998).  
 [6] E. Abrahams, S. V. Kravchenko, and M. P. Sarachik, Rev. Mod. Phys. **73**, 251 (2001).  
 [7] T. Schneider and J. M. Singer, *Phase Transition Approach to High-Temperature Superconductivity* (Imperial College Press, London, 2000); T. Schneider, e-print cond-mat/0104053 (unpublished).  
 [8] M. Wallin, E. Sorensen, S. M. Girvin, and A. P. Young, Phys. Rev. B **49**, 12 115 (1994).  
 [9] M. P. A. Fisher and G. Grinstein, Phys. Rev. Lett. **60**, 208 (1988).  
 [10] S.-K. Ma, Phys. Rev. Lett. **29**, 1311 (1972).  
 [11] J. Ye, Phys. Rev. B **58**, 9450 (1998).  
 [12] G. W. Semenoff, P. Sodano, and Y. S. Wu, Phys. Rev. Lett. **62**, 715 (1989).  
 [13] I. F. Herbut and Z. Tešanović, Phys. Rev. Lett. **76**, 4588 (1996).  
 [14] J. Hove, S. Mo, and A. Sudbo, Phys. Rev. Lett. **85**, 2368 (2000).  
 [15] H. Kleinert and F. S. Nogueira, e-print cond-mat/0104573, and references therein.  
 [16] P. Olsson and S. Teitel, Phys. Rev. Lett. **80**, 1964 (1998).  
 [17] I. F. Herbut, Phys. Rev. Lett. **79**, 3502 (1997); Phys. Rev. B **57**, 13 729 (1998), and references therein.  
 [18] S. N. Dorogovtsev, Phys. Lett. **76A**, 169 (1980); D. Boyanovsky and J. L. Cardy, Phys. Rev. B **26**, 154 (1982); I. D. Lawrie and V. V. Prudnikov, J. Phys. C **17**, 1655 (1984).  
 [19] See R. Mukhopadhyay and P. B. Weichman, Phys. Rev. Lett. **76**, 2977 (1996).  
 [20] Gauge invariance of (1) guarantees preservation of the minimal coupling form in the renormalized theory. See, for example, Ref. [11].  
 [21] B. I. Halperin, T. C. Lubensky, and S.-K. Ma, Phys. Rev. Lett. **32**, 292 (1974).  
 [22] Gauge invariance prohibits a mass term for the gauge field in Eq. (1) that would yield the Yukawa form of the screened interaction in real space.  
 [23] J. Ye and S. Sachdev, Phys. Rev. Lett. **80**, 5409 (1998).  
 [24] S. L. Sondhi and S. A. Kivelson (unpublished).  
 [25] I. F. Herbut, Phys. Rev. Lett. **81**, 3916 (1998); Phys. Rev. B **58**, 971 (1998); **61**, 14 723 (2000).