Maximal Height Scaling of Kinetically Growing Surfaces

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The scaling properties of the maximal height of a growing self-affine surface with a lateral extent L are considered. In the late-time regime its value measured relative to the evolving average height scales like the roughness: $h_L^* \sim L^\alpha$. For large values its distribution obeys $\log P(h_L^*) \sim -A(h_L^*/L^\alpha)^a$. In the early-time regime where the roughness grows as t^β, we find $h_L^* \sim t^\beta[\ln L - (\beta/\alpha)\ln t + C]^{1/b}$, where either $b = a$ or b is the corresponding exponent of the velocity distribution. These properties are derived from scaling and extreme-value arguments. They are corroborated by numerical simulations and supported by exact results for surfaces in 1D with the asymptotic behavior of a Brownian path.

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Kinetic roughening of self-affine surfaces has continuously attracted much interest in the past two decades. It is observed in many systems such as crystal growth, vapor deposition, molecular-beam epitaxy, electrochemical processes, bacterial growth, burning fronts, etc. [1–4]. The investigations mostly dealt with the scaling properties of the surface roughness. Other characteristics, which have been studied more recently, include the distribution of the width [5,6], the distribution of the site-average height velocity in the 1D Kardar-Parisi-Zhang (KPZ) [7] model [8], the density of the extrema [9], the persistence [10], and the cycling effects [11]. In this Letter we focus on another important property: the scaling behavior of the highest point of the surface. In contrast with all characteristics listed above, the maximal height is not a site-average property. Rather, it is an *extremal* one which takes just one single value for every realization of the surface. Since the maximal height is one of the two most *uncommon* values of the height of the whole surface (the second being the lowest height which follows similar behavior), to obtain its scaling behavior and that of its distribution, an inherently different approach is required. It blends together the usual scaling methodology to the roughness with extreme-value statistics of rare events [12].

Besides the theoretical challenge involved in rare event statistics of variables with intricate spatial and temporal properties, the maximal height is of paramount importance in technological applications. It is the most significant property for systems with corroded surfaces in which the deepest, or the weakest, point determines the onset of breakdown [13]. Another example is batteries in which a short circuit occurs when the highest point of the metal surface accumulated on one electrode reaches the opposite one.

In addition to the absolute height measured from the substrate, we define the relative height as the height measured from the evolving average height of the surface. The scaling properties of the maximal height (MH) and the maximal relative height (MRH) are derived from the combined scaling and extreme statistic approaches. They are supported by numerical simulations and by rigorous results from extremal excursion properties of Brownian paths. Our main findings are the following: In the late-time regime the MRH scales as the roughness, and different properties of its distribution are elucidated. In the short-time regime the maximal height grows logarithmically slower than the roughness and has a logarithmic dependence on L (while the roughness is L independent). The exact form of these logarithmic terms depends on the late-time distribution of either the velocity or the MRH.

The surface width $w(L,t)$, where L is the lateral size of the system, is defined as $w^2(L,t) = \overline{H_L(\vec{r},t)^2} - \overline{H_L(t)}^2$. In this definition $H(\vec{r},t)$ is the surface height. An overbar denotes the average over the sites $\vec{r}$ of the basal plane of size L^D $(D = d - 1)$. In particular, $\overline{H_L(t)} = L^{-D}\sum_{\vec{r}\in L^D} H(\vec{r},t)$ is the average height of a given surface at time t, and $\overline{H_L(t)} = vt$ (with v being the growth velocity). We also define the roughness as $W(L,t) = \langle w^2(L,t)\rangle^{1/2}$, where angular brackets denote the average over the ensemble of surface configurations. We recall that it obeys the following scaling form [1,14]: $W(L,t) \sim L^\alpha g[L/\xi(t)]$, where $\xi(t) \sim t^{1/z}$ is the lateral correlation length (z is the dynamic exponent). For large time $t > t_\times(L) \sim L^z$ it reaches its time-independent value, $W_L \sim L^\alpha$, while for $t < t_\times(L)$ it is L independent, $W(t) \sim t^\beta$, where $\beta = \frac{\alpha}{z}$ is the growth exponent. Some generic models of surface growth are random deposition (RD), Edwards-Wilkinson (EW) [15], KPZ [7], DasSarma-Tamborenea (DT) [16], etc.

The relative height is given by

$$h_L(\vec{r},t) = H_L(\vec{r},t) - \overline{H_L(t)}. \tag{1}$$

The height at a site is the sum of the average height and the relative height. If either one fluctuates much more than the other, it will make the dominant contribution to the spread of the height distribution in general and to that of the MH [denoted by $H_L^*(t)$] in particular.

A most important distinction with regard to the MH is, therefore, between a first category of systems in which the velocity v at the large t limit has a nontrivial distribution $P(v)$ (because the instantaneous rate of deposition on the surface is affected by its configuration) and a second category of systems in which $P(v)$ is trivial because the instantaneous deposition rate is independent of the surface configuration. Typical systems of the first category are those in the KPZ universality class [8]. In the second category are, e.g., systems in the EW and DT universality classes with either a constant or a white-noise deposition rate.

The rationale for this categorization will become apparent in the discussion of the early-time regime, which is addressed below following the analysis of the late-time regime. In this latter regime, systems from both categories with $P(v) \neq \delta(v - v_0)$ have their MH distribution dominated by that of the average height itself. Thus their height distributions are given essentially by those of their respective velocities (multiplied by time). For such systems the MRH $h_L^*(\vec{r}, t) = H_L^*(\vec{r}, t) - \overline{H_L(t)}$ describes the subdominant fluctuations and is of interest as well. The MH distribution for some important second category systems in which v is not fluctuating (e.g., electrodeposition at a constant current) is obviously given by that of the MRH, up to a uniform shift by vt.

We begin by analyzing the behavior of the MRH in the late-time regime. In this regime the probability distribution of the relative height is time independent. Consistent with the scaling description, the roughness W_L is the only relevant scale of this distribution. Thus the height distribution depends only on $u = \frac{h}{W_L}$, and $P(u)$ is independent of L [by $P(x)$ throughout we denote the distribution of x, rather than one particular distribution]. Therefore the MRH distribution should depend only on $u^* = h_L^*/W_L$, and be independent of L. This is confirmed numerically for the distributions of the 1D EW model for different L, which are depicted in Fig. 1.

Our simulations indicate that this distribution vanishes for $h_L^* \to 0$, faster than exponential. For large arguments $h_L^* \to \infty$, the distribution also drops faster than exponential and could be fitted to behavior $\sim \exp[\{-A(h_L^*/L^\alpha)^a\}]$ with the exponent $a = 2$. Therefore all moments $\langle (h_L^*/W_L)^k \rangle$ (with k positive or negative) are finite. In particular, both the average and the typical MRH scale like W_L.

While the MRH distribution for this model depends on the boundary conditions (BC), its functional decay at large values does not. We find a behavior $\sim \exp[\{-A(h_L^*/L^\alpha)^a\}]$ with $a = 2$ independent of the BC [17]. (In contrast, the amplitude A varies with the BC.) This is not the case for the behavior of $P(h_L^*/W_L)$ near the origin which is BC sensitive.

These numerical observations are reinforced by analytical results for the 1D Brownian path which faithfully describes the 1D EW and KPZ surfaces in the late-time regime. We represent them as a Brownian path with fixed BC $H(0) = H(L)$ (i.e., as a Brownian bridge), allowing the average height to fluctuate. For the statistical properties of $h_L(r) = H_L(r) - \overline{H_L}$ of this Markovian process, these BC are equivalent to periodic ones.

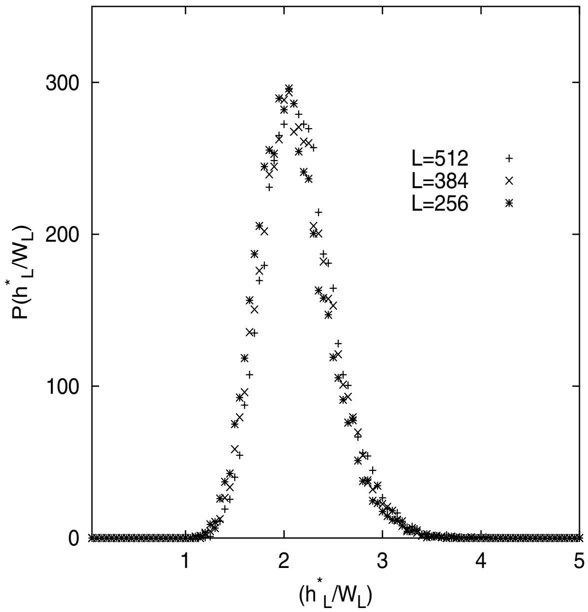

FIG. 1. The distribution of h_L^*/W_L of the 1D EW model with periodic BC for different L. All lengths are expressed in terms of the lattice spacing.

The statistical behavior of the maximal height may be reflected either by the MRH $h_L^* = H_L^* - \overline{H_L}$ or by $\Delta_L = (H_L^* - \overline{H_L})^2$. More complete results are obtained for the distribution of Δ_L. However, h_L^* and Δ_L are simply related by $h_L^{*2} = \Delta_L - w_L^2$.

Averaging this equation over different surface realizations yields

$$\langle h_L^{*2} \rangle = \langle \Delta_L \rangle - W_L^2 \,, \qquad (2)$$

By using path-decomposition techniques [18], we find the Laplace transform (generating function) of the distribution $P(\Delta_L)$:

$$\hat{P}(\lambda) = \langle \exp\{-\lambda \Delta_L\} \rangle = \left[\frac{\sqrt{2\lambda L}}{\sinh(\sqrt{2\lambda L})} \right]^{3/2}. \qquad (3)$$

This immediately implies that $LP(\Delta_L)$ depends only on Δ_L/L. $LP(\Delta_L)$ behaves as $(\frac{L}{\Delta_L})^{5/4} \exp(-\frac{9L}{8\Delta_L})[1 - \frac{5}{8}\sqrt{\frac{\Delta_L}{L}} - \frac{5}{384}\frac{\Delta_L}{L} + \cdots]$ for asymptotically small values of Δ_L/L, and as $(\frac{\Delta_L}{L})^{1/2} \exp(-\frac{\pi^2 \Delta_L}{2L})[1 + c_1(\frac{L}{\Delta_L}) + c_2(\frac{L}{\Delta_L})^2 + \cdots]$ for large values.

Moreover, we find that $\langle \Delta_L \rangle = L/2$. Using the result in Ref. [5], $W_L^2 = L/12$, we deduce from Eq. (2) that $\langle h_L^{*2} \rangle = 5L/12$, confirming the heuristic arguments above. Numerically the distributions of Δ_L and (h_L^{*2}) are very similar through the whole range and have the same exponential decay at large values.

We stipulate that the robustness (with respect to the BC) of the functional decay at large values of the MRH distribution is a general property of correlated surfaces. What is the actual functional decay for a given surface and whether it is related to the distributions of other properties (like that of the height itself, the width, etc.) are left as open questions for future studies. For most of the surfaces we expect this decay to be exponential, characterized by an exponential-tail exponent a. We also speculate that $a = 2$ might hold for any Gaussian surface. Numerical simulations for the DT model [16] in both 1D and 2D and for the KPZ model in 2D clearly show the average MRH to scale as the roughness. The tails of their distributions are exponential and consistent with the value of $a = 2$ [18].

Having considered the late-time regime, we now address the early-time regime. In this regime the correlation length ξ is smaller than L and grows with time. Hence the height variables are uncorrelated if they are separated by a distance $l > \xi$. Thus, it is beneficial to begin by exploring the RD model for which the height variables are totally uncorrelated. We thus consider $h_L(i) = H_L(i) - \overline{H_L(t)}$, $i = 1, 2, \ldots, N$ ($N = L^D$) random uncorrelated variables. Each of them has a binomial distribution which, for large time, converges into a Gaussian distribution of zero mean and variance $W(t) \sim t^{1/2}$. Extreme-value statistics then imply a scaling of the average (or typical) MRH, to leading order, as $W(t)\sqrt{2 \ln N}$ (the distribution of the MRH is the Gumbel distribution [12]). Since the average height $\overline{H_L(t)}$ is essentially uncorrelated with each of the $h(i)$, the MH distribution is that of the MRH shifted by $\overline{H_L(t)}$.

For realistic models with a growing $\xi(t)$, we may imagine dividing the system into $N_\xi = (L/\xi)^D$ cells, such that the surface is correlated within the cells but no correlations exist between $H(\vec{r})$ belonging to different cells. Every cell has its own highest point, and they are also uncorrelated. Thus, we aim to apply again the extreme-value statistics of N_ξ independent variables in order to estimate the MH and the MRH of the whole system. In this regime both have essentially the same distribution up to a trivial shift by the average height. This results from the fact that the MH of a cell has very weak (a power of ξ/L) correlations with the average height of the whole surface.

To proceed with finding their scaling behavior, we need first to find the functional decay of the distribution of the MH within every cell. To that aim, we have to determine which of the cell average heights or the cell MRH is more dominant (i.e., fluctuates more). For systems in the second category their respective fluctuations may be estimated. The fluctuations in the average height of a cell are of order $\sqrt{t}\,\xi^{-D/2} \sim \xi^{z\beta'}$, with $\beta' = (1 - D/z)/2$. The fluctuations have to be compared with the cell MRH fluctuations which scale as the cell roughness: $W_\xi \sim \xi^{z\beta}$. For the linear models $\beta' = \beta/2$, and the MRH fluctuations are dominating. $\beta' < \beta$ holds for all systems in the second category we have checked. We conclude that for such systems the fluctuations in the average height may be discarded and therefore the decay of cell MH distribution is essentially that of the cell's MRH.

Next we need to find the large-value decay of the distribution of the cell MRH. We expect it to follow that of $P(h_L^*/W_L)$ in the late-time regime, with $L = \xi$. $P(h_L^*/W_L)$, however, is only defined for specific BC. Trying to assign such specific BC between the virtual cells is, of course, meaningless. Nevertheless, we will rely on the property, discussed above, that for *all BC this distribution has the same functional decay for large values of its argument.* Therefore, this universal decay with respect to the BC persists even when the BC are ill defined. It follows that $P(h_\xi^*)$ of a single cell MRH is most likely to decay as $\sim \exp[\{-A(h_\xi^*/W_\xi)^a\}]$, and so does the distribution of the MH within every cell (recall that the cells MH and MRH differ by a weakly fluctuating average height).

Extremal-value statistics then predict the highest of the maximal heights of N_ξ cells to scale as $H_L^* \sim W_\xi \times (\ln N_\xi)^{1/a}$. By replacing W_ξ by Bt^β and N_ξ by $[L/t^{\beta/\alpha}]^D$, we obtain [after shifting by the uncorrelated $\overline{H_L(t)}$]

$$\langle h_L^* \rangle = Kt^\beta \left[\ln L - \left(\frac{\beta}{\alpha} \right) \ln t + C \right]^{1/a}, \qquad (4)$$

with nonuniversal constants K and C.

Two conclusions follow: (i) The growth of the MRH is slower than that of the roughness by a logarithmic factor. This is seen in the simulation results in Fig. 2, where we have plotted $(h_L^*/W_L)^2$ vs $\log t$ for the 1D EW model. The plot is consistent with the leading linear behavior predicted by Eq. (4). The deviation from linearity is in the direction expected from the next order correction ($\sim$log-logN_ξ [12]) which becomes increasingly important near the crossover to the late-time regime (as N_ξ becomes smaller). (ii) The MRH depends logarithmically on the system size L for $t < t_\times$ (when the roughness is L independent). This may be observed in Fig. 3 for the same model. Early-time

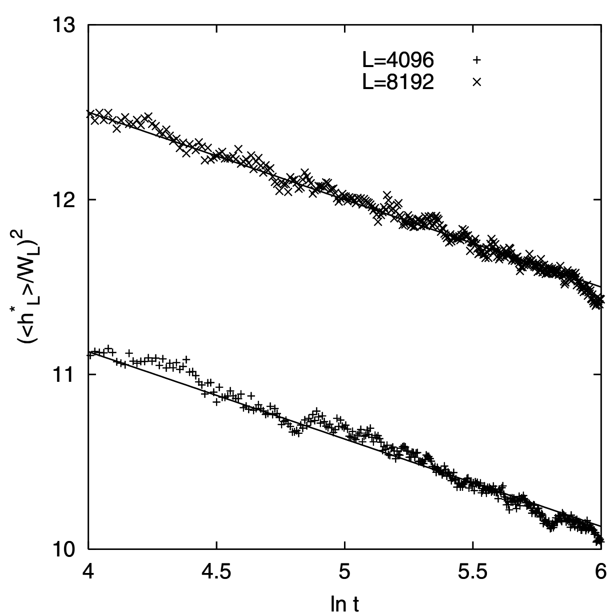

FIG. 2. $[h_L^*(t)/W_L(t)]^2$ vs $\log t$ of the 1D EW model for two values of L. The solid lines are guides for the eyes.

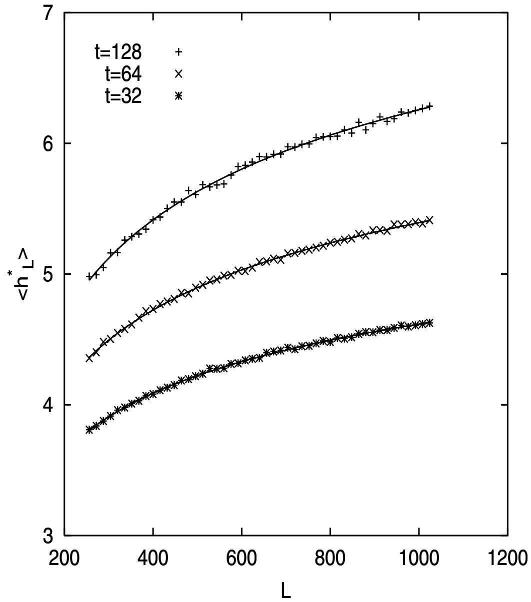

FIG. 3. $h_L^*(t)$ vs L of the 1D EW model for different values of t. The solid lines are fits to $\bar{K}t^{\beta}[\log L + \bar{C}(t)]^{1/2}$.

simulations for the DT model 1D and 2D yield similar behavior [18].

For systems in the first category the early-time behavior might differ. In such systems the possibility of $\beta' > \beta$ within a cell cannot be ruled out (in such a case the growth exponent of the surface assumes the value of β').

This scenario was observed in the 1D KPZ model for which the width of the surface relative height distribution W_ξ was calculated numerically. It was found to behave as $\exp[-(h/W_\xi)^\eta]$, with $\eta = 2$ at the center of the distribution, $\eta_- \simeq 1.6$ in the left tail, and $\eta_+ \simeq 2.4$ for the right tail of largest heights [19]. This behavior mimics very closely that of the velocity distribution in the late-time regime (with $\eta_- = 1.5$ and $\eta_+ = 2.5$ [8]). Thus, the early-time extreme values of the height are determined by *the fluctuations in the average height of a cell, and not from those of the cell's MRH* (in contrast with second category systems discussed above). Therefore, the power $1/a$ of the logarithmic term in Eq. (4) has to be replaced by $1/\eta_+$, at least for $\xi \ll L$. In this regime corrections to scaling might still be important while, for larger values of ξ, N_ξ becomes smaller and the largest typical height moves towards the center of the distribution with possibly an effective exponent η between 2.5 and 2. Similar arguments were used recently [20] to explain the extremal statistics in the energetics of random-bond interfaces.

In summary, we have obtained the scaling behavior of the maximal height. In the short-time regime its growth differs from that of the roughness by a logarithmic correction depending on t and L. The exponent of this correction is determined from the exponential decay of the distribution of either the velocity (first category) or the MRH (second category) in the late-time regime. In this latter regime the surface is correlated throughout the whole system and the average MRH scales like the roughness. It would be desirable if more analytical and numerical results for the MH and MRH (and their distributions) of other models could be obtained. Finally, we hope that experimental systems with accurate roughness scaling over a wide range would exhibit the scaling behavior of the maximal height as well.

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