Laplacian Growth and Diffusion Limited Aggregation: Different Universality Classes

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It had been conjectured that diffusion limited aggregates and Laplacian growth patterns (with small surface tension) are in the same universality class. Using iterated conformal maps we construct a one-parameter family of fractal growth patterns with a continuously varying fractal dimension. This family can be used to bound the dimension of Laplacian growth patterns from below. The bound value is higher than the dimension of diffusion limited aggregates, showing that the two problems belong to two different universality classes.

Laplacian growth patterns are obtained when the boundary Γ of a two-dimensional domain is grown at a rate proportional to the gradient of a Laplacian field P. Outside the domain $\nabla^2 P = 0$, and each point of Γ is advanced at a rate proportional to ∇P [1]. It is well known that without ultraviolet regularization such growth results in finite time singularities [2]. In correspondence with experiments on viscous fingering one usually adds surface tension, or in other words solves the above problem with the boundary condition $P = \sigma \kappa$ where σ is the surface tension and κ the local curvature of Γ [3]. The other boundary condition is that as $r \rightarrow \infty$ the flux is $\nabla P = \text{const} \times \hat{r}/r$. Figure 1 (left) shows a typical Laplacian growth pattern.

Diffusion limited aggregation (DLA) [4] begins with fixing one particle at the center of coordinates in 2 dimensions, and follows the creation of a cluster by releasing fixed size random walkers from infinity, allowing them to walk around until they hit any particle belonging to the cluster. Since the particles are released one by one and may take an arbitrarily long time to hit the cluster, the probability field is quasistationary and in the complement of the cluster we have again $\nabla^2 P = 0$. In this case the boundary condition on the cluster is $P = 0$, but finite time singularities are avoided by having finite size particles. The boundary condition at infinity is exactly as above. A typical DLA is shown in Fig. 1 (right).

In spite of the different ultraviolet regularizations of Laplacian growth and DLA, it was speculated by many authors [5] that the two problems belong to the same "universality class," and it was expected that the resulting fractal patterns will have the same dimension. In this Letter, we argue that this is not the case: there are deep differences

between the two problems, and in particular Laplacian growth patterns have a dimension considerably higher than DLA. In one sentence, the differences between the problems stem from the fact that Laplacian patterns are grown layer by layer, whereas DLA is grown particle by particle. Unfortunately, traditional techniques used to grow Laplacian growth patterns, either numerical [6] or experimental [7], fail to achieve patterns large enough to extract reliable dimensions (see Fig. 9 in [6], for example). The numerical algorithms are extremely time consuming due to the stiffness of the equations involved; experimentally it is difficult to construct large quasi-two-dimensional (Hele-Shaw) cells without introducing serious deformations.

The aim of this Letter is to provide a scheme to simulate the zero surface tension Laplacian growth that has a finite size regularization and thus does not suffer from finite time singularities. We introduce a 1-parameter family of growth processes based on iterated conformal maps [8,9]. Contrary to DLA which grows particle by particle, we will construct the family of growth processes to mimic Laplacian growth, in which a layer is added to the boundary Γ at each growth step, with a width proportional to the gradient of the field. Consider then $\Phi^{(n)}(\omega)$ which conformally maps the exterior of the unit circle $e^{i\theta}$ in the mathematical ω plane onto the complement of the (simply connected) cluster of n particles in the physical z plane. The unit circle is mapped onto the boundary of the cluster. The map $\Phi^{(n)}(\omega)$ is made from compositions of elementary maps $\phi_{\lambda,\theta}$, $\Phi^{(n)}(\omega) = \Phi^{(n-1)}(\phi_{\lambda_n,\theta_n}(\omega))$, where the elementary map $\phi_{\lambda,\theta}$ transforms the unit circle to a circle with a "bump" of linear size $\sqrt{\lambda}$ around the point $\omega = e^{i\theta}$. In this Letter, we employ the elementary map [8]

$$\phi_{\lambda,0}(\omega) = \sqrt{\omega} \left\{ \frac{(1+\lambda)}{2\omega}(1+\omega)\left[1+\omega+\omega\left(1+\frac{1}{\omega^2}-\frac{2}{\omega}\frac{1-\lambda}{1+\lambda}\right)^{1/2}\right]-1\right\}^{1/2} \quad (1)$$

$$\phi_{\lambda,\theta}(\omega) = e^{i\theta}\phi_{\lambda,0}(e^{-i\theta}\omega). \quad (2)$$

With this choice the map $\Phi^{(n)}(\omega)$ adds on a new semicircular bump to the image of the unit circle under $\Phi^{(n-1)}(\omega)$. The bumps in the z plane simulate the accreted particles in the physical space formulation of the growth process. The

recursive dynamics can be represented as iterations of the map $\phi_{\lambda_n,\theta_n}(\omega)$,

$$\Phi^{(n)}(\omega) = \phi_{\lambda_1,\theta_1} \circ \phi_{\lambda_2,\theta_2} \circ \ldots \circ \phi_{\lambda_n,\theta_n}(\omega). \quad (3)$$

With the present technique it is also straightforward to determine the dimension. The conformal map $\Phi^{(n)}(\omega)$
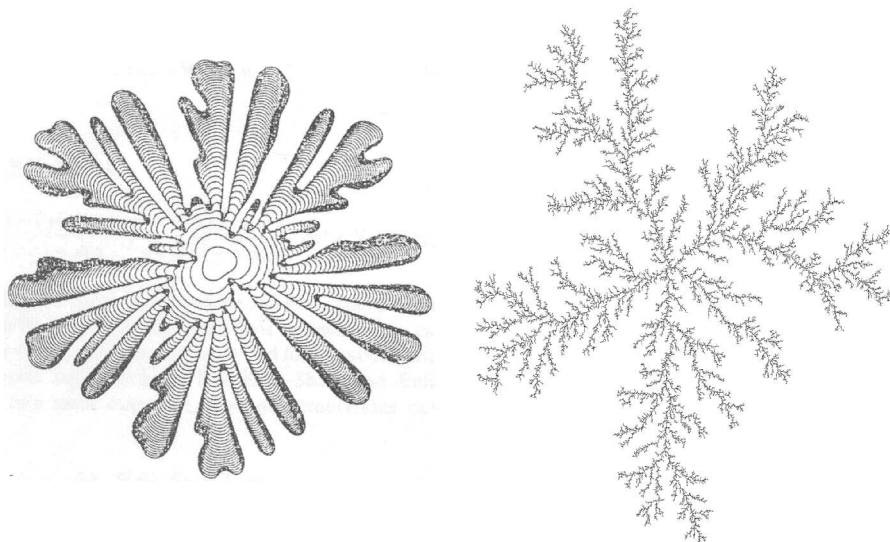
FIG. 1. Left: Typical Laplacian growth pattern with surface tension; cf. Ref. [6]. Right: Typical DLA cluster of 100 000 particles.

admits a Laurent expansion

$$\Phi^{(n)}(\omega) = F_1^{(n)}\omega + F_0^{(n)} + \frac{F_{-1}^{(n)}}{\omega} + \cdots . \qquad (4)$$

The coefficient of the linear term is the Laplace radius, and was shown to scale as

$$F_1^{(n)} \sim S^{1/D}, \qquad (5)$$

where S is the area of the cluster (the sum of the actual areas of the bumps in the physical space). On the other hand, $F_1^{(n)}$ is given analytically by

$$F_1^{(n)} = \prod_{k=1}^{n} \sqrt{(1 + \lambda_k)}, \qquad (6)$$

and therefore can be determined very accurately.

Different growth processes can be constructed by proper choices of the itineraries $\{\theta_i\}_{i=1}^{n}$ [10], and rules for determining the areas of the bumps $\{\lambda_i\}_{i=1}^{n}$. In DLA growth [8,9] one wants to have *fixed size* bumps in the physical space, say of fixed area λ_0. Then one chooses in the nth step

$$\lambda_n = \frac{\lambda_0}{|\Phi^{(n-1)\prime}(e^{i\theta_n})|^2}, \qquad \text{DLA growth.} \qquad (7)$$

The probability to add a particle to the boundary of the DLA cluster is the harmonic measure, which is uniform on the circle. Thus in DLA the itinerary $\{\theta_i\}_{i=1}^{n}$ is random, with uniform probability for θ_i in the interval $[0, 2\pi]$.

For our present purposes we want to grow a *layer* of particles of varying sizes, proportional to the gradient of the field, rather than *one* particle of fixed size. This entails three major changes. First, if we want to grow *one* particle of size proportional to the gradient of the field (i.e., area proportional to $|\Phi^{(n-1)\prime}(e^{i\theta_n})|^{-2}$) we need to choose

$$\lambda_n = \frac{\lambda_0}{|\Phi^{(n-1)\prime}(e^{i\theta_n})|^4}, \qquad \text{present models.} \qquad (8)$$

Second, to grow a layer, we need to accrete many particles without updating the conformal map. In other words, to add a new layer of p particles when the cluster contains m particles, we need to choose p angles on the unit circle $\{\tilde{\theta}_{m+k}\}_{k=1}^{p}$. At these angles we grow bumps which in the physical space are proportional in size to the gradient of the field around the m-particle cluster:

$$\lambda_{m+k} = \frac{\lambda_0}{|\Phi^{(m)\prime}(e^{i\tilde{\theta}_{m+k}})|^4}, \qquad k = 1, 2 \ldots, p . \qquad (9)$$

Last, and very importantly, we need to choose the itinerary $\{\tilde{\theta}_{m+k}\}_{k=1}^{p}$ which defines the layer. This itinerary is chosen to achieve a uniform coverage of the unit circle *before* any growth takes place. The parameter that will distinguish one growth model from another, giving us a 1-parameter control, is the *degree of coverage*. In other words, we introduce the parameter

$$C = \frac{1}{\pi} \sum_{k+1}^{p} \sqrt{\lambda_{m+k}} . \qquad (10)$$

This parameter is the fraction of the unit circle which is covered in each layer, with the limit of Laplacian growth obtained with $C = 1$. It turns out to be rather time consuming to grow fractal patterns with C close to unity. But we will show below that this is hardly necessary; already for C of the order of $1/2$ we will find patterns whose fractal dimension significantly exceeds that of DLA, offering a clear lower bound on the dimension of Laplacian growth patterns.

Once a layer with coverage C had been grown, the field is updated. To do this, we define a series $\{\theta_k\}_{k=1}^{p}$ according to

$$\Phi^{(m)}(e^{i\tilde{\theta}_{m+k}}) \equiv \Phi^{(m+k-1)}(e^{i\theta_{m+k}}). \qquad (11)$$

Next we define the conformal map used in the next layer growth according to

$$\Phi^{(m+p)}(\omega) \equiv \Phi^{(m)} \circ \phi_{\theta_{m+1},\lambda_{m+1}} \circ \ldots \circ \phi_{\theta_{m+p},\lambda_{m+p}}(\omega). \qquad (12)$$

It is important to notice that on the face of it this conformal map appears very similar to the one obtained in DLA,

Eq. (3). But this is deceptive; the distribution of θ values is different, we do not update the map after each particle, and the growth rule is different.

We can achieve a uniform coverage C using various itineraries. One way is to construct the "golden mean trajectory" $\tilde{\theta}_{m+k+1} = \tilde{\theta}_{m+k} + 2\pi\rho$ where $\rho = (\sqrt{5} - 1)/2$. At each step we check whether the newly grown bump may overlap a previous one in the layer. If it does, this growth step is skipped and the orbit continues until a fraction C is covered. Another method is random choices of $\tilde{\theta}_{m+k}$ with the same rule of skipping overlaps. We have tried several other itineraries. Of course, to be an acceptable model of Laplacian growth the resulting cluster should be invariant to the itinerary. This invariance is demonstrated below. The central thesis of this work is that the dimension of the resulting growth patterns is dependent on C only, and not on the itinerary chosen to achieve it. Numerically it is more efficient to use the golden mean itinerary since it avoids as much as possible previously visited regions. In order to achieve comparable growth rates for different layers we inflated λ_0 in Eq. (9) according to $\lambda_0 \to m\lambda_0$ in the layer composed of p particles $\{m + k\}_{k=1}^p$. In Fig. 2 we show F_1 of clusters grown by choosing 3 different itineraries to produce the layers and for two values of C.

We conclude that the dimension (determined by the asymptotic behavior of F_1 vs S) does not depend on the itinerary used to form the layers but on C only.

In Fig. 3 we show three fractal patterns grown with this method, with three different values of C. Even a cursory observation should convince the reader that the dimension of these patterns grows upon increasing C.

In order to calculate the dimension we averaged F_1 of many clusters produced by the golden mean itinerary, each with another random initial angle in each layer. Plots of the averages $\langle F_1 \rangle$ for 3 values of C are presented in Fig. 4.
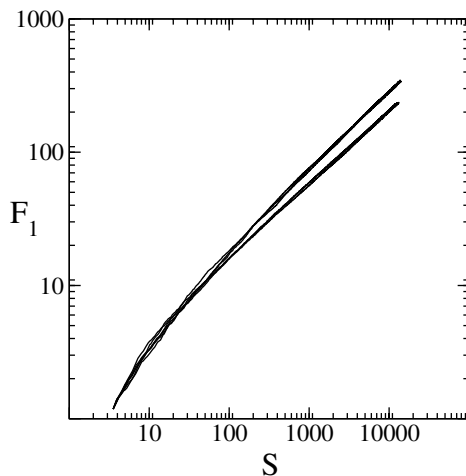
We conclude that the dimension of the growth pattern increases monotonically with C, with $D \approx 1.85$ when $C = 0.6$.

The main point of this analysis is that the dimension of Laplacian growth patterns is bounded from below by the supremum on the dimensions obtained in this family of models. First, Laplacian growth calls for $C = 1$.
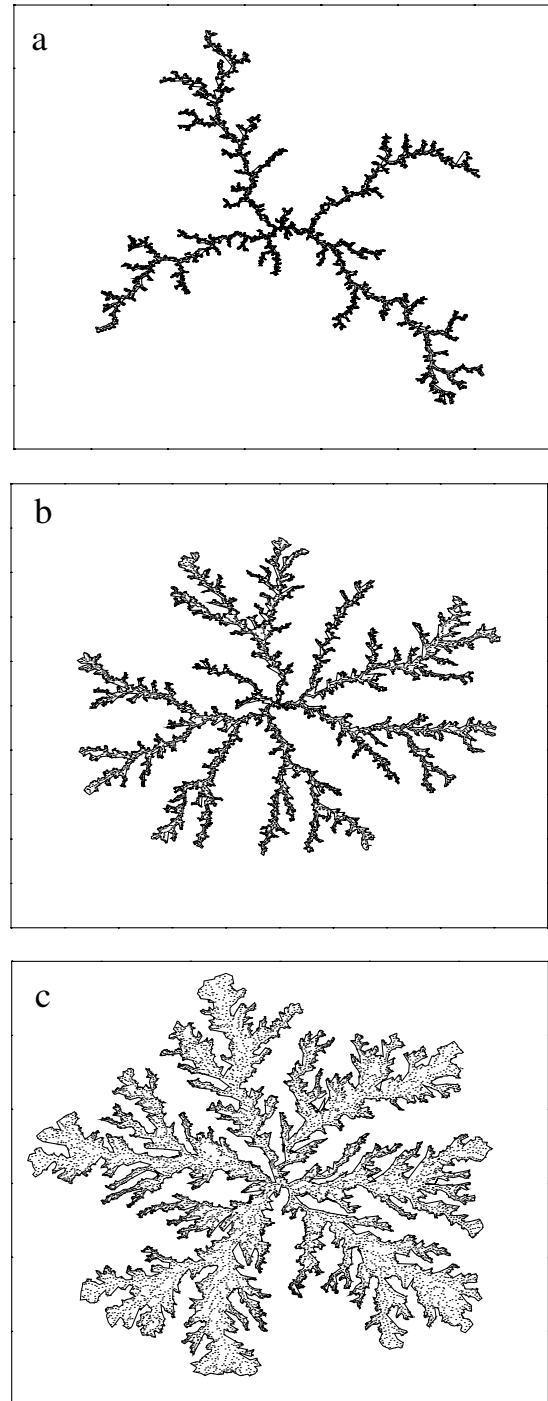

FIG. 2. Log-log plots of F_1 vs S of six individual clusters, using three different itineraries for layer construction, with two values of C. $C = 0.3$ (upper group) and $C = 0.5$ (lower group). Here we use the golden-mean, random, and period doubling itineraries (see Ref. [10]).

FIG. 3. Patterns grown with three different values of C by using the golden-mean itinerary: (a) $C = 0.1$; (b) $C = 0.3$; (c) $C = 0.5$.
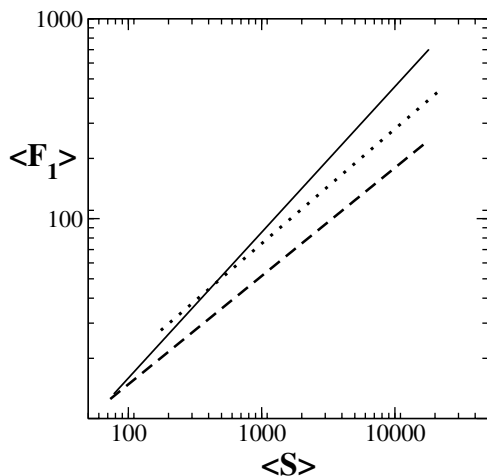
FIG. 4. Linear regressions of log-log plots of $\langle F_1 \rangle$ vs S for three values of C: 0.1 (solid line); 0.3 (dotted line); and 0.6 (dashed line). The slopes of the curves imply dimensions $D = 1.37$, $D = 1.75$, and $D = 1.85$, respectively. The averages are taken over at least 20 clusters.

Second, in Laplacian growth the boundary condition is $P = \sigma \kappa$, suppressing growth at the tips (and relatively favoring growth in the fjords) compared to growth with the boundary condition $P = 0$. Accordingly, on the basis of the results shown in Fig. 4, we propose that the dimension of Laplacian growth patterns exceeds 1.85, putting it distinctively away from the dimension of DLA, which is about 1.71 [11]. We stress that there may be slow crossovers in the dimension estimates, but if these exist they tend to *underestimate* the dimension, leaving our bound intact.

In hindsight, it is difficult to understand how the consensus formed in favor of DLA and Laplacian growth being in the same universality class. Superficially, one could say that in DLA the update of the harmonic measure after each particle is not so crucial, since the effect of such an update is relatively *local* [12]. Thus it may just work that a full layer of particles would be added to the cluster before major interaction between different growth events takes place. However, this view is completely wrong. An incoming random walker lands on top of a previously attached one *very often*. To see this, consider how many angles $\{\theta_j\}$ can be chosen *randomly* on the unit circle before the first overlap between bumps [of linear sizes $\epsilon_j = \sqrt{\lambda_n(e^{i\theta_j})}$]. To get the order of magnitude take $\epsilon_j = \epsilon = \langle \sqrt{\lambda_n} \rangle$. The average number of times that we can choose randomly an angle before the first overlap is $\mathcal{N}(\epsilon) \sim \frac{1}{\sqrt{\epsilon}}$. The length of the unit circle that is covered at that time by the already chosen bumps is $\mathcal{L}(\epsilon) = \epsilon \mathcal{N}(\epsilon) \sim \sqrt{\epsilon}$. It was shown in [9] that for DLA $\langle \lambda_n \rangle \sim \frac{1}{n}$, so that $\epsilon \sim \frac{1}{\sqrt{n}}$, implying $\mathcal{N}(n) \sim n^{1/4}$. Notice that this result means in particular that for a DLA cluster of 1 million particles only less than 50 random walkers can be attached before two of them will arrive at the same site. Moreover, $\mathcal{L}(n) \sim \frac{1}{n^{1/4}} \to 0$ for $n \to \infty$, which means that as the DLA cluster grows, our coverage parameter C goes to zero, rather than to unity

where Laplacian growth is. Taking spatial fluctuations of λ_n into account may change the exact exponents, but not the qualitative result. This argument clarifies the profound difference between growing a whole layer simultaneously and particle by particle. Note, however, that DLA is *not* the $C \to 0$ limit of our 1-parameter family due to the difference between Eqs. (7) and (8).

The results of this study underline once more the delicacy of the issues involved. Fractal patterns depend sensitively on the details of the growth rules. Even though the analytic presentation seems very similar, to the degree that many researchers were led to believe in wide universality classes, we showed here that one must be much more cautious. By lifting the models into *families* of growth patterns depending on a parameter we could demonstrate strong variability of the fractal dimension. Here we constructed the family to bound from below Laplacian growth patterns. A similar family can be constructed to bound DLA from above. This and other aspects of this method will be reported elsewhere.

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