Initial Data for Two Kerr-like Black Holes

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We prove the existence of a family of initial data for the Einstein vacuum equation which can be interpreted as the data for two Kerr-like black holes in an arbitrary location and with spins pointing in arbitrary directions. We also provide a method to compute them. If the mass parameter of one of the black holes is zero, then this family reduces exactly to the Kerr initial data. The existence proof is based on a general property of the Kerr metric which can be used in other constructions as well. Further generalizations are also discussed.

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Introduction.—Black-hole collisions are considered as one of the most important sources of gravitational radiation that may be observable with the gravitational wave detectors currently under construction [1–5]. The first step in the study of black-hole collisions is to provide proper initial data for the Einstein vacuum equations. Initial data for two black holes were first constructed by Misner [6]. Shortly after, Brill and Lindquist [7] studied similar data with a different topology which considerably simplified the construction. Bowen and York [8] included linear and angular momentum. Generalizations of these data were studied in [9–12] (see also the review [13], and references therein).

These families of initial data depend on the mass, the momentum, the spin, and the location of each black hole. When the mass parameter of one of the holes is zero, one obtains initial data for only one black hole. It is physically reasonable to require initial data for one black hole to be stationary, i.e., a slice of Schwarzschild or Kerr spacetime. If this is not the case, it means that spurious gravitational radiation is present in the initial data. In the case of [6] and [7], this limit yields the Schwarzschild initial data. However, when the angular momentum is not zero, one does not obtain the initial data of the Kerr metric.

The Kerr initial data are not included in the families considered above because the restrictions imposed on the conformal 3-metric are too strong. In most cases it is assumed that the conformal metric is flat. However, it appears that the Kerr metric admits no conformally flat slices (in fact, in [14] it has been shown that there does not exist axisymmetric, conformally flat foliations of the Kerr spacetime that smoothly reduce, in the Schwarzschild limit, to slices of constant Schwarzschild time). Weaker conditions on the 3-metric still exclude the Kerr data. For example, in [9] the conformal metric is required to admit a smooth compactification. We will see that, at least for the Boyer-Lindquist slices, this condition is also strong enough to exclude Kerr data.

Recently a number of proposals have been introduced in order to construct more realistic black-hole initial data [15–18]. However, none of these articles provide a rigorous existence proof of the solutions. In some cases, the evidence for the existence of solutions relies on numerical experiments for some values of the parameters. Even if the solution exists for such choices of parameters, it is not clear at all that it will also exist for other choices.

In this paper, we explicitly construct an initial data representing two (or more) Kerr-like black holes in arbitrary locations and with spins pointing in arbitrary directions. When the mass parameter of one of the black holes is zero we obtain exactly the Kerr initial data. A rigorous existence proof is provided assuming some conditions on the parameters. The existence proof is based on general properties of the Kerr initial data which may also be useful in other constructions.

The Kerr initial data.—Consider the Kerr metric in the Boyer-Lindquist coordinates $(t, \tilde r, \theta, \phi)$ [19,20], with mass m and angular momentum per unit mass a such that $m^2 > a^2$. Take any slice $t =$ const. Denote by $\tilde h^k_{ab}$ the intrinsic 3-metric of the slice and by $\tilde K^{ab}_k$ its extrinsic curvature. These slices are maximal, i.e., $\tilde h^k_{ab} \tilde K^{ab}_k = 0$. The metric $\tilde h^k_{ab}$ is given in the coordinates $(\tilde r, \theta, \phi)$ by

$$\tilde h^k \equiv \frac{\Sigma}{\Delta} d\tilde r^2 + \Sigma d\theta^2 + \eta d\phi^2, \tag{1}$$

where

$$\Sigma = \tilde r^2 + a^2 \cos^2\theta, \qquad \Delta = \tilde r^2 + a^2 - 2m\tilde r, \tag{2}$$

and

$$\eta = \sin^2\theta(\Sigma + a^2 \sin^2\theta(1 + \hat\sigma)), \qquad \sigma = \frac{2m\tilde r}{\Sigma}. \tag{3}$$

The metric is singular whenever Δ or Σ vanishes. The zeros of the function Δ are given by

$$\tilde r_+ = m + \delta, \qquad \tilde r_- = m - \delta, \tag{4}$$

with $\delta = \sqrt{m^2 - a^2}$. The extrinsic curvature is given by

$$\tilde K_{k\theta\phi} = N \frac{-2\tilde r m a^3 \cos\theta \sin^3\theta}{\Sigma^2}, \tag{5}$$

$$\tilde{K}_{k\tilde{r}\phi} = N\frac{am\sin^2\theta(-a^4\cos^2\theta + \tilde{r}^2a^2(1+\cos^2\theta) + 3\tilde{r}^4)}{\Delta\Sigma^2}, \tag{6}$$

where

$$N^2 = \frac{\Delta}{\Sigma + a^2\sin^2\theta(1+\hat{\sigma})}. \tag{7}$$

Now, consider the coordinate transformation

$$\tilde{r} = \frac{\alpha^2\cos^2(\psi/2) + \delta^2\sin^2(\psi/2)}{\alpha\sin\psi} + m, \qquad 0 \le \psi \le \pi, \tag{8}$$

where α is a positive constant. This transformation is the composition of the transformation to the quasi-isotropical radius $\bar{r}$ and a stereographic projection, i.e.,

$$\tilde{r} = \bar{r} + m + \frac{\delta^2}{4\bar{r}}, \qquad \bar{r} = \frac{\alpha\cos(\psi/2)}{2\sin(\psi/2)}. \tag{9}$$

It is defined for $\tilde{r} > \tilde{r}_+$, and becomes singular at $\tilde{r}_+$. We consider (ψ,θ,ϕ) as standard coordinates on S^3. The South Pole is given by $\psi = 0$ and the North Pole by $\psi = \pi$; we will denote them by $\{0\}$ and $\{\pi\}$, respectively. Because of the isometry $\bar{r} \to \delta^2/(4\bar{r})$, the transformation (8) maps one copy of the region $\tilde{r} > \tilde{r}_+$ into the region $\psi > \psi_+$ of S^3, where $\psi_+ = 2\arctan(\alpha/\delta)$, and another copy into $\psi < \psi_+$. In the new coordinates, the metric (1) extends to a smooth metric in $S^3 - \{0\} - \{\pi\}$. This manifold defines a spacelike hypersurface in the Kerr spacetime which, in Fig. 28 of [21], corresponds to a horizontal straight line going from one apex of a region I to the opposite apex of the adjacent region I. The poles $\{0\}$ and $\{\pi\}$ are precisely these apexes, they represent the spacelike infinities of the initial data. This hypersurface is a Cauchy surface for an asymptotically flat region of the Kerr spacetime (comprising two regions I and II, respectively).

Using the conformal factor

$$\varphi_k = \frac{\Sigma^{1/4}}{\sqrt{\sin\psi}}, \tag{10}$$

one defines the conformal metric h^k_{ab} by

$$h^k_{ab} = \varphi_k^{-4}\tilde{h}^k_{ab}. \tag{11}$$

The conformal factor φ_k is singular at $\{0\}$ and $\{\pi\}$,

$$\lim_{\psi\to\pi}(\pi - \psi)\varphi_k = \frac{\delta}{\sqrt{\alpha}}, \qquad \lim_{\psi\to 0}\psi\varphi_k = \sqrt{\alpha}. \tag{12}$$

The metric h^k_{ab} has the form

$$h^k_{ab} = h^0_{ab} + a^2 f v_a v_b, \tag{13}$$

where h^0_{ab} is the standard metric of S^3, the smooth vector field v_a is given by $v_a \equiv \sin^2\psi\sin^2\theta(d\phi)_a$, and the function f, which contains the nontrivial part of the metric, is given by

$$f = \frac{(1+\hat{\sigma})}{\Sigma\sin^2\psi}. \tag{14}$$

The function f depends on $a, m, \sin\psi, \cos\theta$. It is smooth in $S^3 - \{0\} - \{\pi\}$. In order to analyze the differentiability of f at the poles, consider a normal coordinate system x^i with respect to the metric h^k_{ab}, centered at one of the poles, and define the radius $|x| = [\sum_{i=1}^3 (x^j)^2]^{1/2}$. In terms of these coordinates the function ψ, given by $\psi = |x|$, is seen to be a C^α function of x^i. From expression (14) one can prove that the function f has the form

$$f = f_1 + f_2\sin^3\psi, \tag{15}$$

where f_1 and f_2 are smooth functions in the neighborhood of the poles, with respect to the coordinates x^i. Since $\sin^3\psi \in W^{4,p}$, $p < 3$ (see, e.g., [22] for the definitions of the Sobolev and Hölder spaces $W^{s,p}$ and $C^{m,\alpha}$), from expression (15) we see that

$$h^k_{ab} \in W^{4,p}(S^3), \qquad p < 3. \tag{16}$$

This is the crucial property of the metric that will be used in the existence proof. In fact, it is the only property of the Kerr metric that we will need. It implies, in particular, that the metric is in $C^{2,\alpha}(S^3)$. Since the poles $\{0\}$ and $\{\pi\}$ are the infinities of the data, the expression (15) characterizes the falloff behavior of the Kerr initial data near spacelike infinity. The Ricci scalar R_k of the metric h^k_{ab} is a continuous function of the parameter a, and for $a = 0$ we have $R_k = 6$, the scalar curvature of h^0_{ab}. Thus, if a is sufficiently small, R_k will be a positive function on S^3. In the following we will assume that the latter condition is satisfied.

The extrinsic curvature of the Kerr initial data remains to be analyzed. Define K^{ab}_k by

$$K^{ab}_k = \varphi_k^{10}\tilde{K}^{ab}_k. \tag{17}$$

The tensor K^{ab}_k is smooth in $S^3 - \{0\} - \{\pi\}$ and at the poles it has the form,

$$K^{ab}_k = K^{ab}_J + Q^{ab}, \tag{18}$$

where $K^{ab}_J = O(|x|^{-3})$. It is trace and divergence free with respect to the flat metric (it contains the angular momentum of the data and the explicit form of this tensor is given in [8]). The tensor Q^{ab} is $O(|x|^{-1})$. If $a = 0$ then $K^{ab}_k = 0$.

The coordinate transformation (8) simplifies considerably if we choose $\alpha = \delta$. This choice makes the metric (13) symmetric with respect to $\psi = \pi/2$. This is useful in explicit calculations. Nevertheless, this choice is not adequate for our present purpose, since it is singular for $\delta = 0$ and one would like to have the flat initial data in this limit. In the following we will assume $\alpha = 1$.

Initial data with two Kerr-like asymptotic ends.—The conformal approach to find solutions of the constraint equations with many asymptotically flat end points i_n is the following [cf. [23,24], and references therein. The setting outlined here, where we have to solve (19) and (20) on the compact manifold has been studied in [9,25,26]]. Let S be a compact manifold (which in our case will be S^3), denote by i_n a finite number of points in S, and define the manifold $\tilde{S}$ by $\tilde{S} = S \setminus \bigcup i_n$. We assume that h_{ab} is a positive definite metric on S, with covariant derivative D_a and positive scalar curvature R. Let K^{ab} be a trace-free symmetric tensor, which satisfies

$$D_a K^{ab} = 0 \text{ on } \tilde{S}. \tag{19}$$

Let φ be a solution of

$$L_h \varphi = -\frac{1}{8} K_{ab} K^{ab} \varphi^{-7} \text{ on } \tilde{S}, \tag{20}$$

where $L_h = D^a D_a - R/8$. Then, the physical fields $(\tilde{h}, \tilde{K})$ defined by $\tilde{h}_{ab} = \varphi^4 h_{ab}$ and $\tilde{K}^{ab} = \varphi^{-10} K^{ab}$ will satisfy the vacuum constraint equations on $\tilde{S}$. To ensure asymptotic flatness of the data at the points i_n we require at each point i_n

$$K^{ab} = O(|x|^{-4}) \quad \text{as } x \to 0, \tag{21}$$

$$\lim_{|x| \to 0} |x| \varphi = c_n, \tag{22}$$

where the c_n are positive constants, and x^i are normal coordinates centered at i_n.

The pair (h^k, K_k) has been obtained from the Kerr solution, and consequently they satisfy Eqs. (19) and (20). The boundary conditions (21) and (22) at each of the poles are also satisfied, since they satisfy Eqs. (12) and (18). The Kerr metric h^k satisfies (16), therefore the coefficients of the elliptic operator L_{h^k} satisfy the hypothesis of the existence theorems proved in [27]. In particular, they are in $C^\alpha(S^3)$. From these theorems, it follows that for an arbitrarily chosen point $i \in S^3$ there exists a unique positive function φ_i, which satisfies

$$L_{h^k} \varphi_i = 0, \text{ in } S^3 - \{i\}, \tag{23}$$

and

$$\lim_{x \to 0} |x| \varphi_i = 1, \tag{24}$$

at i, with $|x|$ denoting the distance from i.

We denote by φ_0, φ_π the solutions so obtained by choosing the point i to be $\{0\}$ and $\{\pi\}$, respectively, and write the Kerr conformal factor φ_k in the form

$$\varphi_k = \varphi_0 + \delta \varphi_\pi + u_k.$$

The function u_k is then in $C^\alpha(S^3)$. The Kerr conformal factor has been decomposed in the "punctures" φ_0, φ_π and the regular part u_k. This procedure will be generalized to include another Kerr-like black hole.

Consider the Kerr initial data in coordinates $(\bar{r}, \theta, \phi)$. Make a rigid rotation such that the spin points in the direction of an arbitrary unit vector S_1^a, and make a shift of the origin $\bar{r} = 0$ to the coordinate position of an arbitrary point i_1. Let the mass and the modulus of angular momentum per unit mass of this data be m_1 and a_1. We apply the stereographic projection (9) and the conformal rescaling (10). In this way, we obtain a rescaled metric $h_{ab}^{k_1} = h_{ab}^0 + a_1^2 f_1 v_a^1 v_b^1$, where f_1 and v_a^1 are obtained from f and v_a by the rotation and the shift of the origin. They depend on the coordinates of the point i_1 and the vector S_1^a. In S^3, this coordinate transformation is a smooth conformal mapping with a fixed point at $\{0\}$. In an analogous way we define the corresponding rescaled extrinsic curvature $K_{k_1}^{ab}$. Take another vector S_2^a and another point i_2 and make the same construction. We define the following metric

$$h_{ab}^{kk} = h_{ab}^0 + a_1^2 f_1 v_a^1 v_b^1 + a_2^2 f_2 v_a^2 v_b^2. \tag{25}$$

By (16) we know that this metric is in $W^{4,p}(S^3)$. It is also clear that for small a_1 and a_2 the scalar curvature is positive. It is assumed that the latter condition is satisfied. This is the only condition in the parameters that we impose in order to prove existence.

Let $\bar{K}_{k_1}^{ab}$ be the trace-free part of $K_{k_1}^{ab}$ with respect to the metric h_{ab}^{kk}. Define the tensor K_{kk}^{ab} by

$$K_{kk}^{ab} = \bar{K}_{k_1}^{ab} + \bar{K}_{k_2}^{ab} + (lw)^{ab}, \tag{26}$$

where $(lw)^{ab}$ is the conformal Killing operator l with respect to the metric h_{ab}^{kk} acting on a vector w^a. In [27] it has been proved that there exists a unique $w^a \in W^{2,p}(S^3)$ such that K_{ab}^{kk} satisfies (19). Whenever a_1 or a_2 is equal to zero, then K_{kk}^{ab} is equal to $K_{k_1}^{ab}$ or $K_{k_2}^{ab}$, respectively, since the solution is unique.

Let φ_0, φ_1, and φ_2 satisfy (23) and (24) for $\{0\}$, $\{i_1\}$, and $\{i_2\}$, respectively, with respect to the metric (25). These solutions exist by the theorem proved in [27]. Let u_{kk} be the solution of the equation

$$L_{h_{kk}} u_{kk} = -\frac{1}{8} K_{kkab} K_{kk}^{ab} \varphi_{kk}^{-7} \text{ on } \tilde{S}, \tag{27}$$

where

$$\varphi_{kk} = \varphi_0 + \delta_1 \varphi_1 \sin(\psi_1/2) + \delta_2 \varphi_2 \sin(\psi_2/2) + u_{kk}, \tag{28}$$

and $\delta_1 = \sqrt{m_1^2 - a_1^2}$, $\delta_2 = \sqrt{m_2^2 - a_2^2}$. Using the existence theorem proved in [27], we know that there exists a unique, positive solution $u_{kk} \in C^\alpha(S^3)$ of Eq. (27). Thus, we have constructed a solution

$$\tilde{h}_{ab}^{kk} = \varphi_{kk}^{-4} h_{ab}^{kk}, \qquad \tilde{K}_{kk}^{ab} = \varphi_{kk}^{10} K_{kk}^{ab}, \tag{29}$$

of the constraint equation. This solution has three asymptotic ends $\{0\}$, $\{i_1\}$, and $\{i_2\}$, and if $a_1 = a_2 = 0$ one obtains the solution given in [7] with mass m_1 and m_2. This is the reason for the factors $\sin(\psi_1/2)$ and $\sin(\psi_2/2)$ in (28).

Then, at least for small a, we expect the same behavior of the apparent horizons as the one discussed there. That is, if the mass parameters m_1 and m_2 are small with respect to the separations between the ends, only two apparent horizons surrounding $\{i_1\}$ and $\{i_2\}$ will appear. This gives a geometric distinction between the ends $\{i_1\}$, $\{i_2\}$ which have an apparent horizon around them, and $\{0\}$ which has not. Whenever the separation of the ends is comparable with the masses, one expects that another apparent horizon appears around $\{0\}$. The evolution of these data will presumably contain an event horizon. The final picture of the whole spacetime will be similar to the familiar "pair of pans" shown in Fig. 60 of [21], which represents a collision and merging of two black holes. Both ends $\{i_1\}$ and $\{i_2\}$ are "Kerr" ends, because whenever $m_1 = a_1 = 0$ we obtain the Kerr initial data, and the same is true for $m_2 = a_2 = 0$. We can expect that the geometry near each of these ends is similar, in some sense, to the geometry of the Kerr initial data when the masses are small with respect to the separation. Numerical comparison for the conformal factor, which exhibits this behavior, has been made in [15] for the axisymmetric case.

If either a_1 or a_2 is equal to zero, we obtain an initial data representing a Schwarzschild and a Kerr black hole. It is important to note that in this case the conformal metric (25) and the conformal extrinsic curvature (26) are exactly the Kerr ones. Then remarkably enough, the only new function needed in order to construct the data in this particular case is the conformal factor φ_{kk}.

Conclusion.—We have constructed a family of initial data that can be interpreted as representing two Kerr black holes. It reduces exactly to the Kerr initial data when the mass of one of them is zero. This is the first rigorous proof of the existence of such a class of initial data. We have chosen the ansatz (25), which is perhaps the simplest one, but other choices are possible too. The only requirement we must impose on the conformal metric (25) is that it reduces to the conformal Kerr metric when a_1 or a_2 is equal to zero and satisfies (16); this is a very mild condition. It is also possible to add an extra term in the extrinsic curvature (26) which contains the linear momentum of each black hole. The existence proof is exactly the same (see [27]). However, we will not recover either Kerr or Schwarzschild data when only one black hole is present, since the Boyer-Lindquist slices are not boosted. This is exactly the same situation as for the boosted data given in [8].

In order to see whether the gravitational waves emitted in the case of our data differ in a significant way from the waves observed for Bowen-York data, it would be interesting to compare the numerical evolution of the corresponding spacetimes.

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