Quantum Phases of Vortices in Rotating Bose-Einstein Condensates

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We investigate the ground states of weakly interacting bosons in a rotating trap as a function of the number of bosons, N, and the average number of vortices, N_V. We identify the filling fraction $\nu \equiv N/N_V$ as the parameter controlling the nature of these states. We present results indicating that, as a function of ν, there is a zero temperature *phase transition* between a triangular vortex lattice phase, and strongly correlated vortex liquid phases. The vortex liquid phases appear to be the Read-Rezayi parafermion states.

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A fundamental characteristic of condensed Bose systems is their response to rotation [1]. A transition to a "normal" phase might be expected at sufficiently high angular velocities, ω, of the container (or trap) by loose analogy with a superconductor in a magnetic field. At zero temperature this phase would constitute a novel *uncondensed* ground state. Such a regime is entered when the vortex cores start to overlap. The corresponding value of ω is unattainable with bulk ^{4}He, but may be achievable in the very dilute degenerate atomic gases initially explored in Ref. [2], and studied extensively in Refs. [3–9]. Apart from the identification [5] of the Laughlin state as the ground state at sufficiently high ω, work on the most interesting regime of large numbers of vortices has been restricted to either mean field theory [3] or exact diagonalization [5–8]. These two approaches have exhibited apparently contradictory pictures. Within Gross-Pitaevskii (GP) mean field theory, the ground states are vortex lattices (distorted by the confinement), with broken rotational symmetry [3]. On the other hand, exact diagonalizations have identified ground states which do *not* have crystalline correlations of vortex locations [5]; they are strongly correlated vortex liquids, closely related to incompressible liquid states responsible for the fractional quantum Hall effect [5–7].

Here we present results of extensive exact diagonalizations (EDs) that elucidate the relationship between these two pictures. By using a periodic geometry, we have been able to study systems containing many vortices up to boson densities far in excess of previous EDs. Our results indicate that both crystalline and liquid phases of vortices exist. A clean distinction between these phases can be made only for a large number of vortices. In this limit, we argue that there is a zero-temperature phase transition as a function of the "filling fraction," $\nu \equiv N/N_V$, the ratio of the number of bosons, N, to the average number of vortices, N_V. For large ν, the ground state is a vortex lattice (characterized by broken translational/rotational symmetry). For small ν the ground states are strongly correlated vortex liquids. We find that the vortex liquid ground states are related to the Read-Rezayi "parafermion" states [10] that were introduced in the context of fractional quantum Hall systems.

In a frame of reference rotating with angular velocity $\omega\hat{z}$, the Hamiltonian for a particle of mass m in an (isotropic) harmonic trap of natural frequency ω_0 is

$$H_\omega = \frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega_0^2\boldsymbol{r}^2 - \omega\hat{z}\cdot\boldsymbol{r}\times\boldsymbol{p}$$
$$= \frac{(\boldsymbol{p} - m\omega\hat{z}\times\boldsymbol{r})^2}{2m} + \frac{1}{2}m[(\omega_0^2 - \omega^2)(x^2+y^2) + \omega_0^2 z^2].$$

The second form indicates the equivalence to the Hamiltonian of a particle of charge q^* experiencing an effective magnetic field $\boldsymbol{B}^* = \nabla\times(m\omega\hat{z}\times\boldsymbol{r}/q^*) = (2m\omega/q^*)\hat{z}$ (the particle also feels a reduced xy confinement). Of particular importance to our discussion is the average filling fraction, ν, for the bosons in this effective magnetic field. For N bosons spread over an area A, one finds

$$\nu \equiv \frac{N}{A}\frac{h}{q^*B^*} = \frac{N}{A}\frac{h}{2m\omega} = \frac{N}{N_V}, \quad (1)$$

where N_V is the *average number of vortices*. For large N_V the vortex density is approximately uniform, and $N_V = (2m\omega A)/h$, or (equivalently) $N_V = 2L/N$ [11], where L is the total angular momentum in units of $\hbar$.

We now introduce repulsive interactions [12]

$$V = g\sum_{i<j=1}^{N}\delta(\boldsymbol{r}_i - \boldsymbol{r}_j), \quad (2)$$

with $g = 4\pi\hbar^2 a/m$, chosen to give the correct s-wave scattering length a. Throughout this work, we make use of the limit of weak interactions formulated in Ref. [2]. For $g \ll \hbar\omega_0\bar{a}^3$, with $\bar{a}$ the interparticle spacing, the bosons are restricted to single-particle states in the lowest Landau level, and lowest oscillator state of z. For $\omega \sim \omega_0$, the repulsive interactions give rise to the appearance of rotating (vortex) ground states [3–9].

GP theory [3] takes account of interactions by finding the fully condensed state that minimizes the total energy.

In EDs [5–7], the ground state is found by diagonalizing (2) within the set of all states of N bosons with fixed total angular momentum L (L is conserved by interactions). An important distinction between these two approaches is that the GP ground states exhibit broken rotational symmetry [3], while the ED ground state is necessarily an eigenstate of angular momentum, L. However, by performing EDs on large numbers of bosons (up to $N = 30$ at $L \sim 2N$), we find that, as N becomes large at fixed L/N, a macroscopic number of quasidegeneracies appear between states with different L. This signals the emergence of broken rotational symmetry. Indeed, it appears from these and other [8,9] studies that as N becomes large for fixed L/N, there is a crossover to a regime in which GP theory is essentially correct. We believe that this crossover is related to the phase transition, discussed in detail below, between vortex liquids at small ν, and a vortex lattice at large ν. Applying a simple Lindemann criterion [13], one finds that a triangular vortex lattice is unstable to quantum fluctuations for $\nu \lesssim 14$. The crossover to GP behavior for increasing N at fixed L/N is the remnant of this phase transition in a system with a finite number of vortices $N_V = 2L/N$.

To investigate in detail the dependence of the ground state on ν, we have conducted extensive (Lanczos) diagonalizations in a *toroidal* geometry [14]. This periodic geometry represents the bulk of a system containing a large number of vortices. We consider a torus of sides a and b. There are then $N_V = (2m\omega ab)/h = ab(q^*B^*/h)$ vortices, which is the number of single-particle states on the torus in the lowest Landau level [14], and hence an integer. Thus, both N and N_V are integers, and $\nu \equiv N/N_V$ is a rational fraction. Finally, we classify all states by the Haldane momentum [15], which runs over a Brillouin zone containing $\bar{N}^2$ points, where $\bar{N}$ is the greatest common divisor of N and N_V. In the following we shall refer to the x and y momenta by the dimensionless vector, (K_x, K_y), using units of $(2\pi\hbar/a)$ and $(2\pi\hbar/b)$. We report only positive values of K_x, K_y up to the Brillouin zone boundary [states at $(\pm K_x, \pm K_y)$ are degenerate by symmetry]. We also choose to measure energies in units of $g/(\sqrt{4\pi}\,\ell^3)$, where $\ell \equiv \sqrt{\hbar/(q^*B^*)} = \sqrt{\hbar/(2m\omega_0)}$ is the magnetic length at $\omega = \omega_0$.

We start by applying Gross-Pitaevskii theory [3] on the torus. In general, the GP ground state is a *vortex lattice,* with broken translational symmetry: the wave function is not an eigenstate of the Haldane momentum, but has weight at a set of reciprocal lattice vectors (RLVs). While the symmetry of the lattice depends, in general, on N_V and the aspect ratio a/b, the absolute minimum of energy is always obtained for a triangular vortex lattice [16].

In ED studies, the ground state is necessarily an eigenstate of the Haldane momentum. The signature of translational symmetry breaking is the development of quasidegenerate levels at the set of momenta given by the RLVs of the broken symmetry lattice [17]. To search for such degeneracies, we show in Fig. 1 the evolution with ν of the excitation energies for $N_V = 8$ vortices at an aspect ratio ($a/b = \sqrt{3}/4$) for which the GP ground state is a triangular lattice.

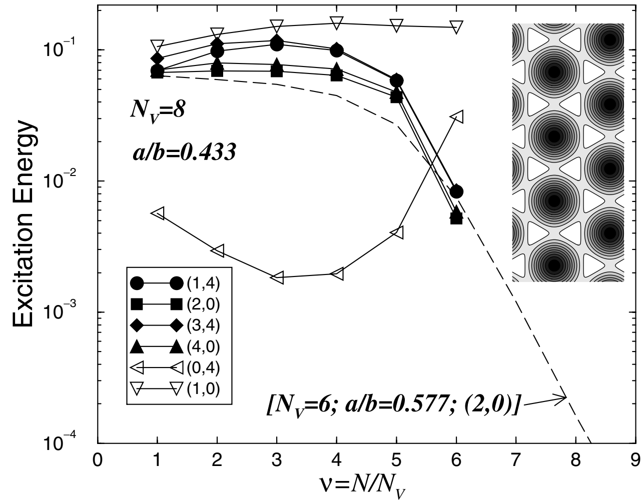

FIG. 1. Solid lines: Excitation energies at momenta measured relative to the ground state, for $N_V = 8$, $a/b = \sqrt{3}/4$ (inset shows the GP ground state: dark = low boson density). The excitation energies at the RLVs of the triangular lattice (filled symbols) collapse at $\nu \sim 6$, signalling the onset of a ground state quasidegeneracy; all other momenta retain nonzero excitation energies (two such momenta are shown as open symbols). Dashed line: The excitation energy at one RLV $(2,0)$ for $N_V = 6$ and $a/b = 1/\sqrt{3}$, showing that the collapse at $\nu \sim 6$ initiates an exponential decrease with ν.

A collapse of the excitation energies at the RLVs of a triangular lattice is observed at $\nu \sim 6$. Similar plots for $N_V = 4, 6$ indicate that the excitation energies at RLVs fall exponentially with ν for $\nu \gtrsim 6$ (shown in Fig. 1 for one RLV for $N_V = 6$). For $\nu = 15$ at $N_V = 6$ the excitation energies are 6 orders of magnitude smaller at the RLVs than at any other momentum. This strong quasidegeneracy at the *reciprocal lattice vectors of the lattice formed in GP theory* indicates a strong tendency to broken translational symmetry [17]. GP theory accurately describes the states at large values of ν.

We view the collapse of the excitation gaps at $\nu \sim 6$ as an indication, in this finite-size system, of a true phase transition from translationally invariant "vortex-liquid" phases, to a (triangular) vortex lattice. The phase transition is rounded due to the finite number of vortices, and becomes sharper for larger N_V (over the range of N_V we can study). Note that we have chosen aspect ratios that are commensurate with a triangular lattice, which is likely to help stabilize the vortex lattice. Similar plots at other aspect ratios show transitions to a vortex lattice at larger values of ν (up to $\nu_c \sim 15$ for $N_V = 4$). One should therefore view $\nu_c \sim 6$ as a *lower bound* on the critical value of ν at which crystallization occurs.

We now turn to discuss the vortex liquids at $\nu \lesssim 6$. In this regime, we find incompressible liquid states similar to those in fractional quantum Hall systems. Some of these incompressible states can be accounted for by the use of a composite fermion construction that has previously been shown to describe accurately ED results on small systems in the disk geometry [6]. In the present uniform geometry, this theory predicts a sequence of incompressible states at $\nu = \frac{\nu_{cf}}{\nu_{cf}+1} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, 2, \infty$, which is a bosonic version of the Jain sequence of fractional quantum Hall states [18]. Many of the strongest incompressible states we find cannot be accounted for in this way. In particular, the largest (finite) value in the composite fermion sequence is $\nu = 2$, while the transition to a vortex lattice does not occur until $\nu = 6$. To investigate the liquid states in this regime, we plot in Fig. 2 the energy gaps as a function of ν for $N_V = 6$ vortices. The energy gap is related to the discontinuity in the chemical potential. To minimize finite-size effects, we define the gap Δ by

$$\Delta(N) \equiv N\left[\frac{E(N+1)}{N+1} + \frac{E(N-1)}{N-1} - 2\frac{E(N)}{N}\right], \quad (3)$$

which reduces to the standard definition as $N \to \infty$.

As well as the Laughlin state at $\nu = \frac{1}{2}$ [5], incompressible states appear clearly in Fig. 2 at $\nu = 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5, 6$ (the loss of gaps for $\nu \gtrsim 6$ is another indication of the transition to the vortex lattice, perhaps reentrant around $\nu = 6$). It is not immediately apparent how to construct incompressible states for this sequence of ν. One possibility is that the vortices themselves are forming Laughlin states. This would provide a set of states with vortex filling fraction $\nu_V = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots$ [19], and hence $\nu = 1/\nu_V = 2, 4, 6, \ldots$. However, this construction does not account for states at $\nu = 1, \frac{3}{2}, \frac{5}{2}, 3, \frac{7}{2}, \frac{9}{2}, 5$ [20]. Moreover, the trial wave functions of this form that we have tested (on a disk) have high interaction energies: while they keep the vortices apart, they do not introduce favorable correlations between the bosons. States of this type are likely to describe systems in which the underlying interactions can be described as repulsive two-body forces between vortices [13,19,21]. They do not provide an accurate description in the present situation, where the interactions cannot be represented by pairwise vortex interactions [22].

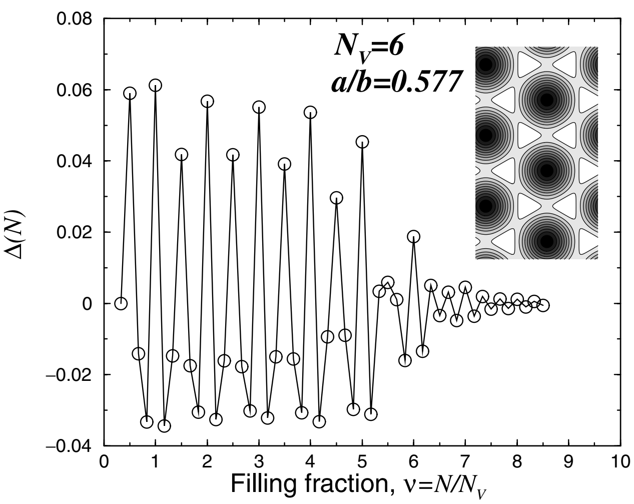

FIG. 2. Energy gap (3) as a function of ν for $N_V = 6$ vortices, at $a/b = 1/\sqrt{3}$. Upward spikes signal values of ν for which the ground state is incompressible. The collapse of the gaps at $\nu \sim 6$ indicates the transition to the vortex lattice phase. (Inset shows the density of the GP ground state.)

A clue to the nature of the incompressible vortex liquid states lies in the existence of an incompressible state at $\nu = 1$, which also cannot be accounted for in terms of noninteracting composite fermions. Rather, this state is well described [23] by the Moore-Read ("Pfaffian") wave function [24]. We find that the exact ground state has large overlap with the Moore-Read state at the Haldane momenta for which it can be constructed on a torus [25].

Motivated by this success, we have compared the incompressible states at higher integer and half-integer ν with parafermion wave functions introduced by Read and Rezayi [10] as generalizations of the Moore-Read state. These states may be represented [26] as a (symmetrized) product of k Laughlin states via

$$\Psi^{(k)}(\{z_i\}) = S\left[\prod_{i<j\in A}^{N/k}(z_i - z_j)^2 \prod_{l<m\in B}^{N/k}(z_l - z_m)^2 \ldots\right], \quad (4)$$

where $z = x + iy$, and we omit the exponential factor of lowest Landau level states as usual. The symbol S indicates symmetrization over all partitions of N particles into sets $A, B, \ldots,$ of N/k particles (we assume that N is divisible by k). The cases $k = 1$ and $k = 2$ correspond to the Laughlin and Moore-Read wave functions. For general k, the wave function (4) describes a system with filling fraction $\nu^{(k)} = N^2/2L = k/2$, and is a zero energy eigenstate of a $(k+1)$-body version of the repulsion (2).

The Read-Rezayi states provide a consistent interpretation of the incompressible states in Fig. 2: they identify the sequence of incompressible states observed in the EDs ($\nu = \frac{k}{2}$ with integer k); they have large overlaps with the exact wave functions, at least up to $\nu = 3$ (the largest ν for which we have made the comparison). We construct the Read-Rezayi states on the torus by diagonalizing the $(k+1)$-body force law directly to find the zero energy eigenstates. In general, we find more than one zero energy eigenstate, and recover a total ground state degeneracy on a torus of $k + 1$, consistent with Ref. [10]. The overlap of the exact ground states of the two-body force (2) with the Read-Rezayi states are given in Table I for $\nu = \frac{k}{2}$. For comparison, the overlaps with the GP ground state are also shown.

In conclusion, we have shown that the ground states of weakly interacting bosons in a rotating trap exhibit both vortex lattices and incompressible vortex liquids. A clear distinction between these phases appears for a large number of bosons, N, and vortices, N_V, and is controlled by the filling fraction $\nu \equiv N/N_V$. Vortex liquid phases appear for $\nu \lesssim \nu_c$ and vortex lattices appear for $\nu \gtrsim \nu_c$. A Lindemann criterion suggests $\nu_c \sim 14$, while exact diagonalizations indicate $\nu_c \sim 6$. Current experiments [27] with $N \sim 10^5$ and $N_V \sim 10$ are deep in the regime in

TABLE I. Wave function comparisons of the exact ground states of the two-body force law at $\nu = k/2$, for $N_V = 6$ and $a/b = 1/\sqrt{3}$. In each case we report the following: the Haldane momenta at which Read-Rezayi states exist, with degeneracies; the overlap of the exact ground state with the Read-Rezayi state (where there is more than one such state, we report the total overlap within this set); the overlap of the exact ground state with the GP ground state (we first project the GP state onto each component of momentum; N/W indicates that the GP ground state has no weight at this momentum).

k	ν	$(K_x, K_y) \times$ degeneracy	$\lvert\langle\Psi^{(k)} \mid \Psi\rangle\rvert$	$\lvert\langle\Psi^{GP} \mid \Psi\rangle\rvert$
1	1/2 (Laughlin)	$(0,0) \times 2$	1.000	0.555
2	1 (Moore-Read)	$(3,3) \times 1$	0.989	N/W
2	1 (Moore-Read)	$(3,0) \times 1$	0.982	0.408
2	1 (Moore-Read)	$(0,3) \times 1$	0.981	0.493
3	3/2	$(0,0) \times 4$	0.967	0.234
4	2	$(0,0) \times 2$	0.956	0.242
4	2	$(3,0) \times 1$	0.966	N/W
4	2	$(0,3) \times 1$	0.935	N/W
4	2	$(3,3) \times 1$	0.844	0.547
5	5/2	$(0,0) \times 6$	0.956	0.163
6	3	$(3,3) \times 2$	0.960	N/W
6	3	$(3,0) \times 2$	0.944	0.198
6	3	$(0,3) \times 2$	0.744	0.534
6	3	$(0,0) \times 1$	0.852	N/W

which the ground state is a vortex lattice. Experiments that access the quantum-melted vortex liquid phases will require specific attention to small system sizes and high angular momentum. Our results indicate that novel correlated states emerge in this regime, which are well described by the Read-Rezayi parafermion states whose excitations obey non-Abelian statistics [10].

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