Dynamic Patterns and Self-Knotting of a Driven Hanging Chain

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When shaken vertically, a hanging chain displays a startling variety of distinct behaviors. We find experimentally that instabilities occur in tonguelike bands of parameter space, to swinging or rotating pendular motion, or to chaotic states. Mathematically, the dynamics are described by a nonlinear wave equation. A linear stability analysis predicts instabilities within the well-known resonance tongues; their boundaries agree very well with experiment. Full simulations of the 3D dynamics reproduce and elucidate many aspects of the experiment. The chain is also observed to tie knots in itself, some quite complex. This is beyond the reach of the current analysis and simulations.

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Spatially extended dissipative systems driven out of equilibrium give rise to fascinatingly complex dynamics in which the interplay of multiple spatial and temporal scales is a central feature; examples range from chemical pattern formation to fluid turbulence [1]. Mathematical descriptions of these systems are usually posed in terms of partial differential equations (PDEs), and often one spatial dimension (1D) is sufficient to capture some of the characteristic complexity [2]. An important class of truly 1D problems involves the motion of lines or filaments embedded in 3D space, such as vortex lines in a type-II superconductor, defect lines in liquid crystals, line vortices in inviscid fluids, or polymers in imposed flows. Topological aspects, such as twisting, breaking, reconnecting, or even knotting, then become an essential part of the dynamics. Simple mechanical systems such as strings or chains can display many of these same aspects, including singularity formation [3], as well as peculiarities such as self-collisions and inextensibility. Here we focus on a completely classical problem from mechanics: a hanging chain driven at its support.

The statics and dynamics of flexible filaments are among the classic problems in PDEs [4], including the catenary shape of a hanging chain and the vibrational modes of a taut string [5]. The mathematical treatment of this continuum problem dates back to Euler, Lagrange, and Bernoulli [4,6]. New dynamics are possible when one or both of the ends are driven in time [7], which injects energy into the system; similar behaviors are found with a flexible rod [8,9]. When surrounded by a fluid, the dynamics may be dominated by viscous drag [10], or coupled strongly to the fluid's inertia [11].

In this Letter we present an experimental and theoretical study of the dynamics of a many-linked chain which hangs freely from a vertically oscillating support. To our knowledge, the richness of this simple system has not yet been explored. We find that the dynamics organize into several distinct states, which include nonlinear pendular motions, precessional states, and apparently chaotic motions. The presence of chaos is not unexpected, as our system is essentially a chain of N coupled pendula, a well-known chaotic system even for $N = 2$ [12]. We mathematically model the chain as a flexible, inextensible string, which is itself described by a highly nonlinear PDE. This system exhibits a classical parametric linear instability, giving good quantitative agreement with the observed transitions between states of motion. Numerical simulations of the full PDE capture much of the nonlinear dynamics of the chain with remarkable fidelity. Finally, we show that the dynamics can result in self-knotting.

Experimental observations.—The possibility of complex motion for a flexible filament is easily demonstrated by shaking a piece of (cooked) spaghetti; quickly one perceives that certain frequencies produce very unstable motion. In our study we use a thin stainless steel chain made of beads of diameter $d = 2.4$ mm, maximum interbead spacing $\ell = 3.4$ mm, and linear mass density $\rho = 0.095$ g/cm. The chain is slightly compressible with a minimum spacing of 2.7 mm (20% of ℓ). The chain is driven sinusoidally by vertically oscillating the upper end with a motor, at frequencies f from 1 to 4 Hz and amplitudes $A = 1.3$, 2.0, and 2.7 cm. The ratio of driving to gravitational acceleration g, $\Gamma \equiv A(2\pi f)^2/g$, ranges from 0.1 to 1.6.

To explore this system, we vary the driving frequency f at fixed amplitude A, with chain lengths L ranging from 7 to 80 cm. We observe several transitions between distinct dynamic states (see Fig. 1). The simplest is a rodlike motion where the chain appears almost rigid as it moves up and down (not shown). As f is increased, an instability occurs resulting in a planar *pendular* motion, in which the chain swings back and forth at half the driving frequency (Fig. 1a). At higher frequencies we observe a wildly energetic motion (Fig. 1b), during which the chain explores its surrounding space in a chaotic fashion, occasionally limited by collisions with itself. For different L or A these same transitions are seen at different frequencies. By introducing the nondimensional driving amplitude $\varepsilon \equiv A/L$ and frequency $\omega \equiv 2\pi f\sqrt{L/g}$, the transition frequencies collapse onto continuous curves in (ε, ω) parameter space,

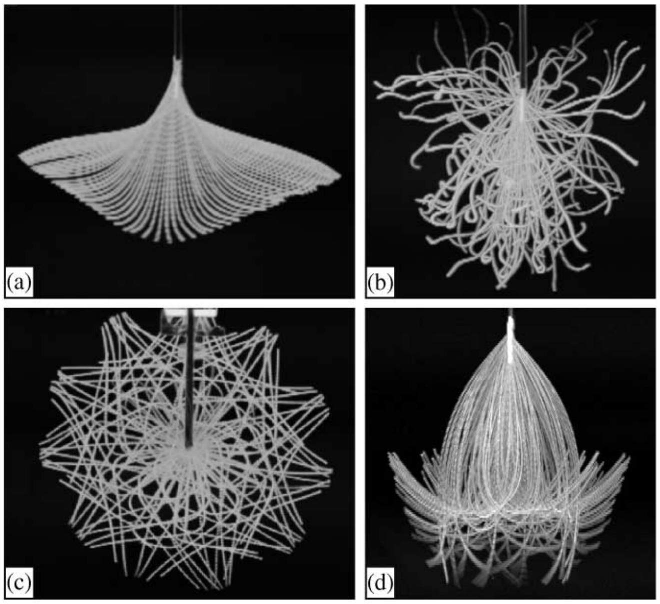

FIG. 1. Collage of the chain position at successive times in different dynamic regimes ($L = 22.7$ cm): (a) pendular swinging ($f = 2.7$ Hz, side view); (b) chaotic ($f = 2.3$ Hz, side view); rotational stellate pattern ($L = 36.4$ cm, $f = 3.5$ Hz): (c) view from below; (d) side view. In (a),(b), 4 s of data are shown, with $\Delta t = 1/30$ s; in (c),(d), many rotation periods are shown, with $\Delta t = 1/15$ s.

as shown in Fig. 2a, which are suggestive of resonance tongues [13]. We do not find hysteresis in the transitions; they occur at the same driving frequency to within 1%.

We also observe a state, occurring only at large L (small ε), in which the chain simultaneously rotates and swings three dimensionally around its suspension point, which we term *stellate* as the chain traces a multipointed pattern during its motion (Figs. 1c and 1d). The rotation can occur in either a clockwise or a counterclockwise direction, with a longer period than the swinging (by a factor of about 4 in Figs. 1c and 1d). As with the pendular state, the chain end is at its most elevated position when the suspension point is at the bottom of its stroke.

A nonlinear chain model.—Mathematically, we treat the chain as a flexible, inextensible string moving according to Newton's laws, with forces given by gravity and axial tension T. The dynamics of the chain position $\mathbf{X}(s,t)$ (s is arclength) are governed by the PDE

$$\rho \mathbf{X}_{tt} = [T(s,t)\mathbf{X}_s]_s - \rho g \hat{\mathbf{z}} + \mu \mathbf{X}_{tss} \tag{1}$$

for $s \in (0, L)$, with boundary conditions (i) $\mathbf{X}(0,t) = A \cos 2\pi f t\, \hat{\mathbf{z}}$ (driving at the fixed end) and (ii) $T(L,t) = 0$ (no forces act on the free end). Here subscripts refer to partial differentiation, thus $\mathbf{X}_s$ is the unit tangent. The slight friction due to the relative rotation of beads along the chain is modeled by a viscous damping term with coefficient μ. The tension is determined by the constraint that the string is inextensible, or that infinitesimal material lengths are invariant: $(\partial/\partial t)\mathbf{X}_s^2 = 2\mathbf{X}_s \cdot \mathbf{X}_{ts} = 0$. Differentiating again yields $\mathbf{X}_s \cdot \mathbf{X}_{tts} = -\mathbf{X}_{st}^2$, or

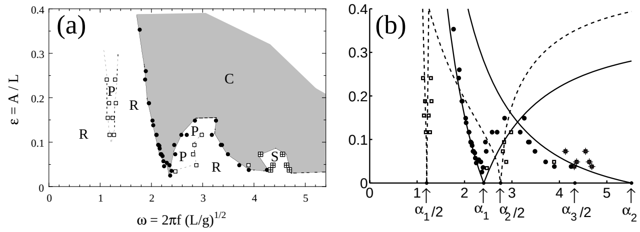

FIG. 2. (a) Nondimensional plot of amplitude vs frequency showing regions of different dynamic behavior for the experiment: rodlike (R); chaotic (C); pendular (P); stellate (S); (b) an overlay of the data with the predicted resonant tongues, with labelled $\omega_c = \alpha_m/n$. Dashed lines: $n = 2$ tongues (pendular motion), solid lines: $n = 1$ tongues. Note the proximity of the stellate region to the $m = 3, n = 2$ tongue (not shown).

$$\mathbf{X}_s \cdot [T(s,t)\mathbf{X}_s]_{ss} = -\rho \mathbf{X}_{ts}^2 - \mu \mathbf{X}_s \cdot \mathbf{X}_{tsss}, \tag{2}$$

a linear boundary value problem for T. Rewriting these equations in dimensionless form, one is left with three parameters: two in the boundary condition $\mathbf{X}(0,t) = \varepsilon \cos\omega t$, and one in the damping term $\nu \mathbf{X}_{tss}$ (we take ν to be small, 10^{-3}).

Several comments on Eqs. (1) and (2) are in order. First, these equations can be derived from the continuum limit of a large number of mass-bearing beads connected by rigid, massless rods. Second, in the absence of driving and damping ($\varepsilon = \nu = 0$) the sum of kinetic and potential energies, $\mathcal{E} = \frac{1}{2}\int_0^1 \mathbf{X}_t^2\, ds + \int_0^1 \mathbf{X} \cdot \hat{\mathbf{z}}\, ds$, is conserved. Third, in the absence of damping, Eq. (1) is a nonlinear wave equation that is hyperbolic when $T > 0$, that is, when the string is under extension. However, the dynamics do not seem to preclude *a priori* compressive, negative tension whose appearance could produce ill-posed evolution. Finally, unlike the physical chain, this model happily allows $\mathbf{X}(s,t)$ to pass through itself. While the inclusion of contact forces is possible, we find a remarkable degree of agreement without this complication.

We first consider the linear behavior of small displacements from a straight chain oscillating vertically. A straightforward generalization of the classical analysis for a hanging chain expresses the solution of the linearized problem (with $\nu = 0$) as a Bessel series, with each amplitude C_m governed by a Mathieu equation:

$$\frac{d^2 C_m}{dt^2} + \frac{\alpha_m^2}{4}(1 + \varepsilon\omega^2 \cos\omega t)C_m = 0, \tag{3}$$

where $m = 1, 2, 3, \ldots$, and α_m is the mth root of J_0. The behavior of Mathieu equations is well understood [14]: for small ε, one finds that for each mode m there exists an infinite number of parametrically unstable driving frequencies $\omega_c = \alpha_m/n$, which branch out into tongues of parametric instability in the (ε, ω) plane [13]. Figure 2b shows the $n = 1, 2$ tongues of the $m = 1, 2$ modes, superimposed with the experimentally determined transition boundaries.

With no fitting parameters, the agreement is excellent, especially for the lowest mode ($m = 1$).

Using numerical simulation, we have also investigated the fully nonlinear behavior of Eqs. (1) and (2). Beginning from initial data for which the chain is nearly vertical, Fig. 3a shows the long-time behavior of the chain model for the same forcing parameters as Fig. 1a. The similarity is striking. Moreover, as seen in the experiment, the upswing and the downswing of the chain are essentially indistinguishable; the chain transits nearly the same shapes on the way up as on the way down. Also, $T > 0$ at all times. A directly related point, illustrated in Fig. 3b, is the vanishing of the kinetic energy at each oscillation, when the suspension point is at the bottom of its stroke, and the string end is in its most elevated position. That is, the entire chain, with its many degrees of freedom, is following a time-periodic solution in which it is momentarily at rest. This is an infinite dimensional analog of "brake orbits"—periodic orbits with points of zero kinetic energy—whose nature and existence are studied in low-dimensional Hamiltonian systems [15].

Figure 3c shows the results of simulation in the chaotic regime. At least in its complexity it bears a strong resemblance to the experiment (cf. Fig. 1b). One difference is that the distribution of positions in the simulation has a near up/down symmetry not apparent in the observations. This may be due to losses of energy in the experimental chain as it collides with itself. Still, the simulation suggests that the dynamics in this parameter region are intrinsically complex. Figure 3d shows kinetic energy over many cycles of the forcing. Because the initial position is rodlike, the kinetic energy is initially small, but increases over several cycles to approach a quasiperiodic fluctuation about a mean ~ 1. There are intermittent decreases in the energy, which correspond to collapses of the string back towards the rodlike state, as also observed in the experiment. Also, while there are isolated instances for which the tension becomes briefly negative somewhere along the chain, the dynamics arranges itself so that the string is almost always under extension.

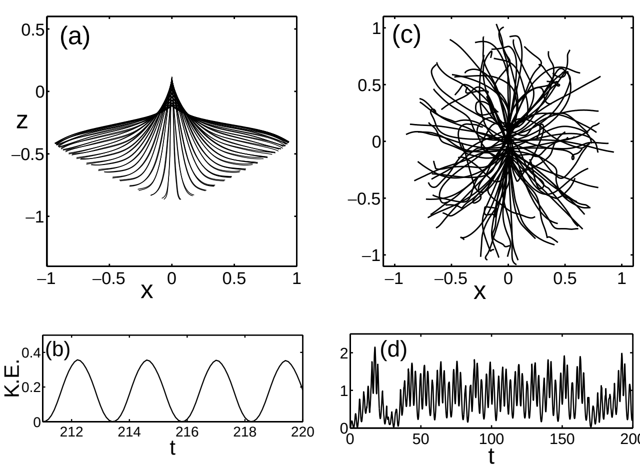

FIG. 3. Simulated dynamics of Eqs. (1) and (2): (a) Side view snapshots of the pendular state, $(\varepsilon, \omega) = (0.12, 2.62)$; (b) kinetic energy for the same state; (c) 100 side view snapshots at equally spaced times in the chaotic state, $(\varepsilon, \omega) = (0.12, 2.16)$; (d) kinetic energy for the same state.

Figure 4 shows a long-time simulation yielding stellate dynamics. Again, the similarity with experimental observation is quite remarkable, including the hatlike shape when viewed from the side (cf. Fig. 1d). This simulation uses the lowest amplitude and frequency producing stellate dynamics in the experiment. At the parameters used for Figs. 1c and 1d we find numerically that the stellate state appears frequently but intermittently with rapid losses of energy and collapses to nearly rodlike dynamics. It may be that lengthier simulations are required to relax into a stellate "attractor." It is also possible that there are aspects of the experiment not represented in our model.

Self-tying of knots.—It seems to be common knowledge among sailors and parlor magicians that a freely hanging rope can tie knots in itself when shaken. Thus it might come as no surprise that we observe knotting in our experiment. The self-tying of knots occurs in the chaotic regime, and the knots vary widely in their complexity (Fig. 5). In addition to modifying the motion, knots are also known to affect the distribution of tension [16]. Immediately after a knot ties, the chain loses nearly all kinetic energy and stabilizes into a rodlike motion, with the knot resting close to the free end. The most frequently observed knot is the figure eight (Fig. 5b), and not the simpler trefoil (Fig. 5a) [17]. The figure-eight knot occurs regularly in certain regions of parameter space, for example, every 50 min on average ($\sim$7000 periods) at $(\varepsilon, \omega) = (0.12, 2.1)$.

To investigate the striking absence of the simple trefoil knot, we "artificially" tied one into the chain while it was at rest. When the chain was oscillated, the trefoil rapidly slipped off the free end and untied; when tied 10 cm from the free end of a 28 cm chain, this takes about 10 s. This explains the apparent rarity of self-tied trefoils. In fact, it is probable that this simplest knot is tying and then untying throughout the chaotic dynamics [18].

This simple experiment and its theoretical model provoke a number of questions. What is the role of the

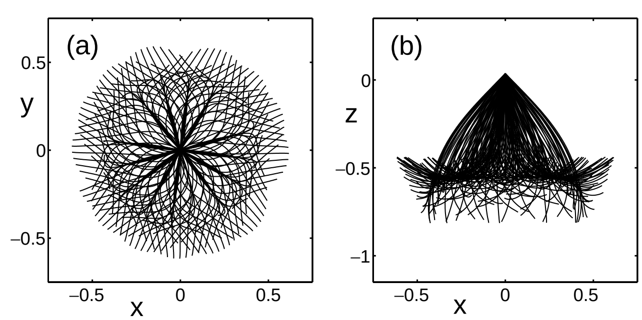

FIG. 4. 200 snapshots of the simulated stellate dynamics, $(\varepsilon, \omega) = (0.037, 4.23)$: (a) viewed from below; (b) side view.

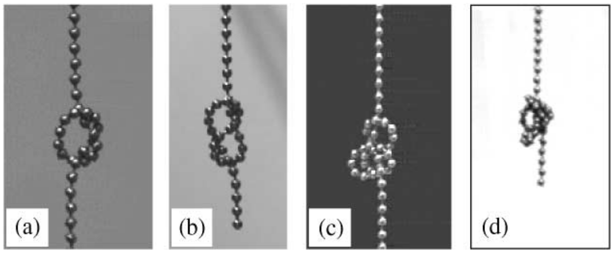

FIG. 5. Self-tied knots observed in the driven chain: (a) trefoil; (b) figure eight (most frequently observed); (c) and (d) are more complex knots. Note that the complex knots do *not* decompose into combinations of simpler knots.

nonlinear pendular and stellate states in more complex dynamics? Within the intermittencies we have observed, it is clear that the oscillatory states can act as coherent structures around which more complicated dynamics revolve. Related studies have begun in simpler PDE wave systems exhibiting spatial and temporal chaos [19]. Another area of inquiry concerns the nature of solutions to our model PDE. These solutions develop and support shocks (here somewhat smoothed by damping), and for closed, undamped strings there is evidence for finite-time curvature singularities tied to points of zero tension [3]. Such singularities are most likely important to interpreting the observations reported here, especially in the chaotic regimes where collapses to rodlike motion occur.

Collisions are another fascinating aspect of the system. Here the physicality of the chain exceeds its elegant mathematical description. In the chaotic regime, collisions dissipate energy but also prevent the chain from visiting regions of its phase space. Further, it is likely that the self-tying of knots involves collisions [18]. This may be relevant to polymer physics; like a thermally driven polymer, the chaotic chain explores its phase space under the constraints of inextensibility and no self-crossing. Knots are known to be present in polymers, and may even play a functional role in the biological polymer DNA [20–22]. There have been many recent direct visual observations of single polymer molecules such as actin [23] and DNA [24,25]. A detailed study of a mechanical system may shed light on analogous questions for polymers which are difficult to resolve on microscopic scales, such as chain configurations and knot dynamics [26,27].

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[1] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[2] P. Collet and J-P. Eckmann, *Instabilities and Fronts in Extended Systems* (Princeton University, Princeton, NJ, 1990).
[3] A. Thess, O. Zikanov, and A. Nepomnyashchy, Phys. Rev. E **59**, 3637 (1999).
[4] S. S. Antman, *Nonlinear Problems of Elasticity* (Springer, New York, 1995).
[5] Lord Rayleigh, *The Theory of Sound* (Dover, New York, 1945).
[6] D. Bernoulli, Comm. Acad. Sci. Petrop. **6**, 108 (1738); J. Cannon and S. Dostrovsky, *The Evolution of Dynamics: Vibration Theory from 1687 to 1742* (Springer, New York, 1981).
[7] S. Lubkin and J. J. Stoker, Q. Appl. Math. **1**, 215 (1943).
[8] D. J. Acheson and T. Mullin, Nature (London) **366**, 215 (1993).
[9] A. R. Champneys and W. B. Fraser, Proc. R. Soc. London A **456**, 553 (2000).
[10] R. E. Goldstein, T. R. Powers, and C. H. Wiggins, Phys. Rev. Lett. **80**, 5232 (1998).
[11] J. Zhang, S. Childress, A. Libchaber, and M. Shelley, Nature (London) **408**, 835 (2000).
[12] T. Shinbrot, C. Grebogi, J. Wisdom, and J. T. Yorke, Am. J. Phys. **60**, 491 (1993).
[13] J. Kevorkian and J. D. Cole, *Perturbation Methods in Applied Mathematics* (Springer, New York, 1981).
[14] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer, New York, 1983).
[15] See, for example, A. Weinstein, Ann. Math. **108**, 507 (1978), and references therein.
[16] J. H. Maddocks and J. B. Keller, SIAM J. Appl. Math. **47**, 1185 (1987).
[17] Note that this is not a knot in the strict sense, but more technically a "cut knot" due to the free end; see G. Burde and H. Zieschang, *Knots* (De Gruyter, Berlin, 1985).
[18] A. Belmonte and S. T. Eldakar (to be published).
[19] See D. Cai and D. W. McLaughlin, J. Math. Phys. (N.Y.) **41**, 4125 (2000), references therein.
[20] S. Wasserman and N. Cozzarelli, Science **232**, 951 (1986).
[21] D. Sumners, Math. Intell. **12**, 71 (1990).
[22] A. Stasiak, V. Katritch, J. Bednar, D. Michoud, and J. Dubochet, Nature (London) **384**, 122 (1996).
[23] A. Ott, M. Magnasco, A. Simon, and A. Libchaber, Phys. Rev. E **48**, 1642 (1993).
[24] T. T. Perkins, D. E. Smith, and S. Chu, Science **264**, 819 (1994).
[25] B. Maier and J. Radler, Phys. Rev. Lett. **82**, 1911 (1999).
[26] Y-J. Sheng, P-Y. Lai, and H-K. Tsao, Phys. Rev. E **58**, 1222 (1998).
[27] Recently a mechanical system similar to ours (involving a chain on an oscillating planar surface) has been used for just such a study; see E. Ben-Naim, Z. A. Daya, P. Vorobieff, and R. Ecke, Phys. Rev. Lett. **86**, 1414 (2001).