Depinning with Dynamic Stress Overshoots: Mean Field Theory

J. M. Schwarz and Daniel S. Fisher
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
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An infinite-range model of an elastic manifold pulled through a random potential by a force F is analyzed focusing on inertial effects. When the inertial parameter M is small, there is a continuous depinning transition from a small-F static phase to a large-F moving phase. When M is increased to M_c, a novel tricritical point occurs. For $M > M_c$, the depinning transition becomes discontinuous with hysteresis. Yet, the distribution of discrete "avalanche"-like events as the force is increased in the static phase for $M > M_c$ has an unusual mixture of first-order-like and critical features. The results may be relevant for the onset of crack propagation and for dynamics of geological faults.

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A wide variety of driven systems are characterized by an elastic manifold that is pulled through a quenched random medium by a uniform applied force [1]. If the dynamics is dissipative—e.g., sliding charge density waves [2], vortex lattices in superconductors [3], and domain walls in ferromagnets [4]—such systems exhibit a critical depinning transition in the absence of thermal fluctuations. A small applied force F is not enough to overcome the random pinning forces and the system remains trapped in one of many possible metastable configurations. However, as F is slowly increased, some sections become unstable and move, only to be stopped by the elastic forces from more strongly pinned neighboring regions. As F is increased even further, there will be a sequence of these discrete, localized "avalanche" events with a distribution of sizes s [5]. As a critical force F_c is approached from below, arbitrarily large avalanches can occur. Above F_c, the driving force is able to overcome the pinning, giving rise to a nonzero average velocity $\bar{v}$. The transition between the two phases is continuous with $\bar{v}$ playing the role of an order parameter that vanishes as $\bar{v} \sim (F - F_c)^\beta$ as F_c is approached from above. This class of dissipative systems has been analyzed by renormalization group (RG) methods [1] with the results supported by numerical [6] and limited experimental [7] evidence.

In some systems, in particular crack fronts in brittle materials [8], geological faults [9], and motion of contact lines of droplets on dirty or rough surfaces [10], the dynamics are not dissipative and inertial effects can be important. In this Letter, we explore the effects of inertial stress transfer in which the motion of one segment creates a transient stress, in addition to the static elastic stress, on other segments. In the presence of pinning, the motion will be mostly forward; this makes the positive (forward pulling) parts of the transient stress transfer the most important; we call these *stress overshoots*. Although we will focus only on such dynamic stress transfer effects, direct effects of inertia will have similar consequences: if a segment moves forward in an underdamped manner so that it overshoots—passing a new local minimum before relaxing back into it—the stress it transfers to the other regions will reflect this overshoot.

Our focus will be on the nature of the depinning transition in the presence of stress overshoots; a key question is whether or not the transition from the pinned to the moving phase remains continuous or becomes "first order"—characterized by a finite jump in $\bar{v}$ as a function of the applied force and perhaps by hysteresis. We will also consider the distribution of avalanche sizes as the depinning transition is approached from below, of particular relevance for geological faults as these represent the "earthquakes."

Near the depinning transition, the motion will be very jerky with a lot of starting and stopping. Thus, the essential physics can be captured by models in which space, time, and the manifold's displacement in the direction of motion, $h(\mathbf{x}, t)$, are all discrete. The elastic stress on a segment $\mathbf{x}$ takes the general form

$$\sigma(\mathbf{x}, t) = \sum_{\mathbf{y}} \sum_{\tau \geq 0} J_{\mathbf{xy}}(\tau) h(\mathbf{y}, t - \tau) - \hat{J} h(\mathbf{x}, t), \quad (1)$$

with the static stress transfer given by $J^s_{\mathbf{xy}} = \sum_\tau J_{\mathbf{xy}}(\tau)$ and $\hat{J} = \sum_{\mathbf{y}} J^s_{\mathbf{xy}}$. We take the simplest form of the stress overshoots: only lasting for one time step. With nearest neighbor coupling we have $J_{\mathbf{xy}}(0) = \frac{(1+M)}{Z}$ and $J_{\mathbf{xy}}(1) = -\frac{M}{Z}$ for $\mathbf{x} \neq \mathbf{y}$ where Z is the number of nearest neighbors and M is the magnitude of the stress overshoot. For $M \leq 0$, a key property of the elastic stress transfer is that it is monotonic, i.e., increasing in time when the h's increase. This implies that the average velocity and hence the critical force are unique [11]. This monotonicity is an essential feature for the RG analysis [2]. For $M > 0$, the stress transfer is *nonmonotonic* in time so that the average velocity is not necessarily a unique function of F.

We take the displacements $h(\mathbf{x}, t)$ to be pinned at a discrete set of possible values with associated pinning (yield) strengths $f_Y[\mathbf{x}, h(\mathbf{x}, t)]$. The dynamics is simple: if the total force on segment $\mathbf{x}$ exceeds f_Y at that point, the segment jumps forward, i.e., if $\sigma(\mathbf{x}, t) + F > f_Y[\mathbf{x}, h(\mathbf{x}, t)]$ then $h(\mathbf{x}, t+1) = h(\mathbf{x}, t) + \Delta[\mathbf{x}, h(\mathbf{x}, t)]$ with Δ the distance to the next pinning position; otherwise $h(\mathbf{x}, t+1) = h(\mathbf{x}, t)$. As long as the jumps $\Delta[\mathbf{x}, h(\mathbf{x}, t)]$ are random variables independently drawn from a smooth distribution $D(\Delta)d\Delta$, the randomness in the pinning strengths is not essential and, for simplicity, we make them all equal.

We now take the first step in analyzing such systems by studying a mean field—more precisely an infinite-range—model. For the case without stress overshoots, such a model was the needed starting point for the RG analysis of finite-dimensional systems. We thus consider a model of N segments each coupled to all the others. The stress on $h(x, t)$ is then simply

$$\sigma(x, t) = \bar{h}(t) - h(x, t) + M(\bar{h}(t) - \bar{h}(t-1)), \quad (2)$$

where $\bar{h}(t) \equiv \frac{1}{N}\sum_y h(y, t)$ is the average over the other segments.

In this infinite range model, all sites are statistically equivalent and coupled only via the mean field $\bar{h}(t)$. We can therefore fully characterize the configuration by the excess forces on the segments: $f_t(x) \equiv \sigma(x, t) + F - f_Y$ and study the the equation of motion for the excess force distribution $\rho_t(f_t)df_t$. A self-consistency condition follows from the "spatial" averaging of the excess force:

$$\langle f_t \rangle = \int_{-\infty}^{\infty} f_t \rho_t(f_t)\, df_t = F - f_Y + M\bar{v}_t, \quad (3)$$

where $\bar{v}_t \equiv \bar{h}(t) - \bar{h}(t-1)$ is the instantaneous spatially averaged velocity. For static solutions, $\bar{v} = 0$, the stress overshoot plays no role and one can show that there are many solutions as long as the applied force is less than the static critical force $F_{c0} = f_Y - \frac{1}{2}\frac{\overline{\Delta^2}}{\bar{\Delta}}$, where the bars here denote averages over $D(\Delta)$.

We first consider the moving phase, studying the *steady state* limit for which $\bar{v}_t = \bar{v} > 0$ is independent of time. Equation (3) implies that in this case the stress overshoot has the same effect as an additional applied force, $F \rightarrow F + M\bar{v}$, and the steady state $\rho(f)$ depends only on $\bar{v}$ (and D). It is easiest to work with the particular jump distribution $D(\Delta) = e^{-\Delta}$ which corresponds to random pinning positions with density $1/\bar{\Delta}$ chosen to be unity. However, the qualitative behavior is similar for any smooth pinning distribution with $D(0)$ finite and nonzero. For M less than a critical value, $M_c = 1$, $\bar{v}$ rises continuously for $F > F_{c0}$ as

$$\bar{v} \sim \frac{F - F_{c0}}{M_c - M} \quad (4)$$

for small $\bar{v}$. The linear increase in $\bar{v}$ for mean-field models with jumps is the average displacement of a segment responds linearly in a nonsingular way to an increase in the *total* force on it [1]. For the exponentially decaying jump distribution, the full expression for $\bar{v}$ is

$$\bar{v} = \frac{-\epsilon_0 + \mu + \sqrt{(\epsilon_0 + \mu)^2 + 2\epsilon_0}}{1 + 2\mu}, \quad (5)$$

with $\epsilon_o \equiv F - F_{c0}$ and $\mu \equiv M - M_c$. Note that for large applied force $\bar{v}$ saturates in a generally unphysical manner.

As M increases to M_c, the width of the continuous depinning transition shrinks to zero and there is a "tricritical" point at $M = M_c$, $F = F_{c0}$. Near this tricritical point the scaling behavior is $\bar{v} \sim |\mu| \sim \sqrt{\epsilon_0}$, indicating a new universality class of depinning transitions.

For $M > M_c$, there is a *jump* in $\bar{v}$ from $\bar{v}_{\min} \sim \mu$ to zero when F is decreased through a new *lower critical force* $F_c^{\downarrow} = F_{c0} - (1 + \mu - \sqrt{1 + 2\mu})$. Just above $F_c^{\downarrow}$ there are two stable solutions: a static one and a moving one with $\frac{d\bar{v}}{dF} < 0$. Between $F_c^{\downarrow}$ and F_{c0} there is hysteresis as might be expected at a first-order-like phase transition: it is harder to stop the motion in the presence of overshoots than in their absence, but once the overall motion does stop, the overshoots will have less effect as F is increased back up again. A schematic phase diagram is shown in Fig. 1.

We now investigate the avalanche dynamics below the depinning threshold. An avalanche is the forward motion of a finite number of segments in response to the initial motion of one segment. The applied force is increased only in order to trigger the motion of the initial segment and is then held fixed until the avalanche stops. The segments that move forward in a single finite avalanche are those whose f is within an infinitesimal, $\mathcal{O}(\frac{1}{N})$, slice near $f = 0$ of the distribution $\rho(f)$; for continuous $\rho(f)$ all that matters is

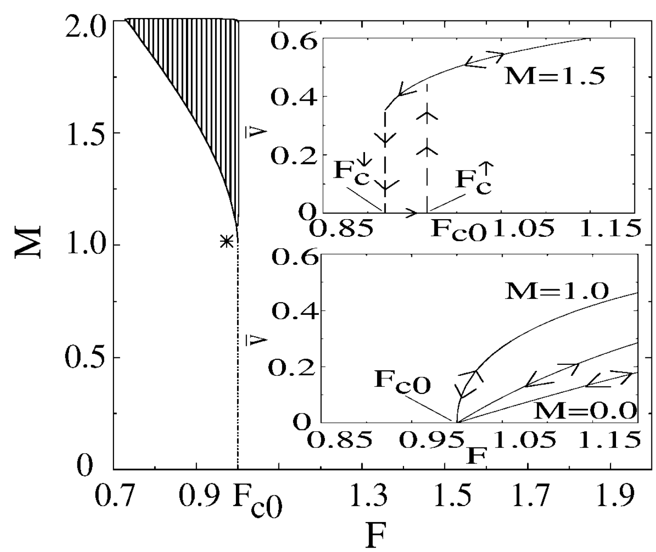

FIG. 1. Schematic phase diagram as a function of F and M with $f_Y = 2.0$. The dashed phase boundary at F_{c0} is a critical line separating the stationary and the moving phases. Above the tricritical point (*) depinning is hysteretic with vertical lines indicating the region of metastability; the critical force on decreasing F, $F_c^{\downarrow}$ is always the solid line, but the critical force on increasing F, $F_c^{\uparrow}$, is history dependent. Insets: $\bar{v}(F)$ for $M \leq M_c$ (lower) and $M > M_c$ (upper); arrows indicate the direction of change of F.

thus $\rho(0)$. The total number of segments that hop forward at one time step, n_t, will have Poisson statistics with mean determined by the increase in the total force on a segment from the previous time step. The mean number that hop at a given time t is given by

$$\langle n_t \rangle = \rho(0)\bar{\Delta}[(1 + M)n_{t-1} - M n_{t-2}]. \tag{6}$$

If this mean becomes negative, the avalanche will stop. We define the parameter, $\Gamma \equiv \rho(0)\bar{\Delta}$—essentially the mean local response to an increase in the total force.

We are primarily interested in the large avalanches, for which n_t will typically be large. Given large n_{t-2} and n_{t-1}, the randomness will cause approximately Gaussian fluctuations in n_t of magnitude of order $\sqrt{n_t}$. With no stress overshoots, $M = 0$, we can approximate the dynamics by

$$\frac{dn(t)}{dt} \simeq (\Gamma - 1)n(t) + \sqrt{n(t)}\,\eta(t), \tag{7}$$

with η white noise. The constraint that $n(t)$ be positive makes finding the distribution of the avalanche size, $s \equiv \int dt\, n_t$, a first-return-to-the-origin problem. A critical point occurs at $\Gamma = 1$ $[\rho_c(0) = \frac{1}{\bar{\Delta}}]$, at which there is a power law distribution of avalanche sizes, $\mathrm{Prob}(ds) \sim \frac{1}{s^{3/2}} ds$ for large s [1,12]. For $\Gamma > 1$, infinite avalanches occur as the motion of one segment spawns an average of more than one descendant at the next time step. For smaller Γ, the cutoff in the avalanche size distribution is given by $\tilde{s} \sim \frac{1}{(1-\Gamma)^2}$ and $\mathrm{Prob}(ds) \sim \frac{1}{s^{3/2}} e^{-(1-\Gamma)^2 s/2} ds$ for large s as shown in Fig. 2.

In the more general case with stress overshoots, the stochastic equation becomes second order in time since the number of segments that hop forward is dependent on two previous time steps. With a *small* amount of stress overshoot, the avalanche dynamics is strongly overdamped with critical point still at $\Gamma = 1$, but the distribution of avalanche sizes is modified by a coefficient $1/(1 - M)$.

Near the tricritical point, we have

$$\Gamma M \frac{d^2 n(t)}{dt^2} \approx (\Gamma - 1)n(t) + (\Gamma M - 1)\frac{dn(t)}{dt} + \sqrt{n(t)}\,\eta(t). \tag{8}$$

At the critical value of M, $M_c = 1$, the avalanche dynamics becomes undamped giving rise to a new distribution of avalanche sizes. At the tricritical point $\frac{d^2 n(t)}{dt^2} = \sqrt{n(t)}\,\eta(t)$, and we have a "particle" with random acceleration whose first return to the origin we are interested in. The tricritical distribution of avalanche sizes is found to broaden to $\mathrm{Prob}(ds) \sim \frac{1}{s^{5/4}} ds$ for large s [13]. The typical avalanche of size s now lasts for a time $\tau \sim s^{1/4}$ in contrast to $\tau \sim s^{1/2}$ in the absence of stress overshoots.

For $M > M_c$, the deterministic part of the avalanche equation of motion is that of a time-reversed harmonic oscillator. If $\rho(0)$ is small—i.e., below the critically damped line—$n(t)$ grows exponentially with time, but then crashes down to zero as a result of the incipient oscillation.

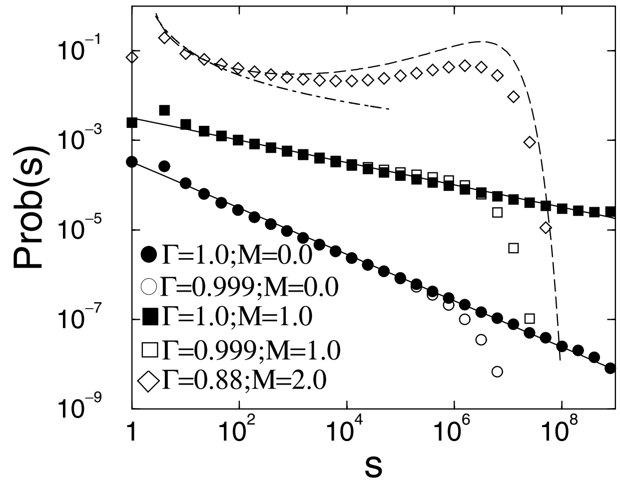

FIG. 2. Log-log plot of avalanche distribution: $\mathrm{Prob}[s/\sqrt{2} <$ size $< \sqrt{2}s]$. The solid lines have slope $\frac{1}{2}$ and $\frac{1}{4}$, the theoretical predictions at criticality, $M < 1$, and tricriticality, $M = 1$. The dashed curve is from the deterministic approximation when $M = 2$, $\Gamma = 0.88$; this is just below the "first-order" line at which the dot-dashed curve obtains. The symbols denote numerical data.

Above a critical Γ, $\Gamma_c(M) = \frac{4M}{(1+M)^2}$ [from Eq. (6)], large avalanches grow exponentially and become infinite. On the critical line, some avalanches become infinite while others remain finite. Because of the exponential growth of $n(t)$ both above and below the critical line, the randomness will not have much of an effect once an avalanche becomes large and a deterministic analysis in which all the randomness is encoded into random initial n and $\frac{dn}{dt}$ becomes valid. Just below the critical line, there is a clear delineation in the distribution of avalanche sizes between those avalanches that would remain finite and those that would become infinite at the critical line. Indeed, just below $\Gamma_c(M)$, a finite fraction of the avalanches will have size of order $\hat{s} \sim \exp[A(M)/\sqrt{(\Gamma_c(M) - \Gamma)}]$ to within a multiplicative factor of order unity; these last roughly half the oscillation period. Note that for $M \gtrsim 1$, $A \sim \mu \equiv M - 1$ and the scaling of $1 - \Gamma$ versus μ is like that of $F - F_{c0}$ versus μ in the moving phase. Most of the other avalanches will be small, but with a long tail in $\log s$ scaling as

$$\mathrm{Prob}(ds) \sim \frac{1}{s(\ln s)^2} ds = \frac{d(\ln s)}{(\ln s)^2} \tag{9}$$

for $1 \ll \ln s \ll \ln \hat{s}$. This broad distribution for the finite avalanches persists at the depinning transition, but now there is also a nonzero probability that an avalanche will be infinite as indicated in Fig. 2. These results suggest a new hybrid transition for $M > M_c$ with a diverging scale, $\hat{s}$, but also discontinuities, particularly in $\bar{v}(F)$.

So far, we have not established the connection between the driving force, F, and the density of about-to-jump segments $\rho(0)$. This is needed to connect together the analysis of the pinned and the moving phases. As long as the avalanches remain finite and the driving force is adiabatically increased, the total force distribution $\rho(f)df$

will be independent of M; the avalanche distribution will only affect *when* segments jump and hence the small scale ($f \sim \frac{1}{N}$) behavior of the stationary configurations between avalanches. For generic initial conditions—continuous $\rho(f)$ which must be consistent with randomly positioned pinning values of $h(x,t)$—Γ will increase with F to reach unity only right at F_{c0}, although the way in which it approaches this value is nonuniversal and history dependent. Thus for $M < 1$, $\Gamma_c = 1$ and the infinite avalanches will only occur when $F > F_{c0}$, the same point above which $\bar{v} > 0$.

For $M > 1$, however, Γ will reach the value $\Gamma_c(M)$ at a value of the driving force $F_c^{\uparrow}$, that depends on the initial conditions as well as on M. From an analysis of Eq. (3), it can be shown that $F_c^{\uparrow}$ is between $F_c^{\downarrow}(M)$, the minimum F for which a moving solution exists, and F_{c0}, as should be expected. Thus, if F is increased above $F_c^{\uparrow}$ and back down there will be hysteresis. If F is decreased below $F_c^{\downarrow}$ so that the system stops and is then increased back up again, the behavior can be complicated as $\rho(f)$ will generally *not* be continuous. Depending on the parameters [including $D(\Delta)$] and the history, the depinning transition on the second increase of F can either be of the hybrid type discussed above with an exponentially growing characteristic size $\hat{s}$, of large avalanches which diverges at a *new* $F_c^{\uparrow}$; or a strongly discontinuous transition with Γ jumping from below Γ_c to above discontinuously without an extensive number of precursor large avalanches. Some of this behavior is summarized in Fig. 1.

We have found that in a simple infinite-range model of depinning with dynamic stress overshoots, subtle behavior can occur. For large overshoots, a hybrid transition exists when the force is increased adiabatically. This is characterized by exponentially large, almost-deterministic avalanches followed by runaway at a critical force to a nonzero velocity state; this moving state persists to a *lower* critical force as the force is decreased back down. Various history dependent effects can occur in this large overshoot regime. For smaller stress overshoots, there is a reversible critical transition similar to that in the absence of overshoots. These two regimes are separated by a "tricritical" point which represents a new universality class.

All of this behavior should obtain for a broad class of infinite-range models, including ones with locally underdamped relaxation caused by inertia [14,15]. But the crucial question is: what of this behavior persists in finite-dimensional models? For large stress overshoots the depinning is surely of a different character than in purely dissipative systems, although what aspects of it might resemble the infinite-range model is far from clear. For small stress overshoots, it has been argued elsewhere that the critical force with $M > 0$ is generically *less* than F_{c0} in finite-dimensional models with long-range interactions [8]. But whether this implies that the transition immediately changes character when *any* stress overshoots are present—and thus presumably that there is no tricritical point—or whether dissipativelike critical behavior can still exist with small overshoots, we must leave as an open question. This is particularly relevant for the case of geological faults as the two types of behavior found in our mean-field model could be thought of, if the applied force is provided by an appropriate weak slowly pulled spring, as representing power-law Gutenberg-Richter distributions of earthquakes versus "characteristic earthquake" behavior with only a few small earthquakes interspersed by large fault-spanning events [16]. Preliminary indications from numerical studies of finite dimensional versions of this model suggest that both of these types of behavior are possible.

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