On the Validity of the Independent Hot-Spot Model

Philippe Mounaix
Centre de Physique Théorique, UMR 7644 du CNRS, Ecole Polytechnique, 91128 Palaiseau Cedex, France
(Received 27 April 2001; published 7 August 2001)

The results of the independent hot-spot (IHS) model are compared to those of the underlying stochastic amplifier in the regime where the coupling of the amplifier is close to its critical value. The considered case is that of a 1D linear amplifier with at most one hot spot per interaction length. It is shown that the validity of the critical coupling given by the IHS model depends on the correlation function of the pump field and should be discussed in each particular case. The discrepancy between the IHS model and the underlying amplifier is shown to be due to the random fluctuations of the hot-spot field around its dominant, deterministic, component.

DOI: 10.1103/PhysRevLett.87.085006 PACS numbers: 52.38.–r, 52.35.Mw

Much experimental and theoretical work has been devoted over the last two decades to studying the influence of laser beam smoothing on scattering instabilities [1]. In the case of spatially smoothed beams, such as random phase plate beams [2], a good idealized model of the physics into play can be obtained by regarding the laser-plasma system as a stochastic convective amplifier driven by the modulus square of a Gaussian field. Unfortunately, even in the simplest linear limit of this model very few exact analytical results have been obtained so far [3,4]. Among them, the most important is the concept of critical intensity defined as the average laser intensity at which the average *linear* reflectivity diverges [5], which corresponds experimentally to a threshold of the instability. Regarding the outcome of a given experiment, it has been shown in Ref. [4] that this divergence can be related to a transition from a below-critical regime, where the backscattered power is uniformly distributed over the amplifier cross section, to the above-critical regime, where it is highly nonuniformly distributed. This result can be used as an alternative definition of the critical intensity in real systems (and in more realistic nonlinear models) in which there is no divergence of the average reflectivity.

In order to deal with scattering instabilities of smoothed beams near and above the critical intensity, a simplified version of the stochastic convective amplifier model, the so-called "independent hot-spot model" (or IHS model), has been developed [5]. In this model, the macroscopic reflectivity of the plasma is assumed to be mainly determined by the high overintensities (or hot spots) of the laser field, randomly distributed in the interaction region. For the sake of completeness, it should be noticed that beside the laser-plasma interaction context, models in which the physics is determined by the occurrence of rare and intense hot spots are also relevant to the interaction of a smoothed laser beam with other nonlinear media like, e.g., liquids and crystals. As examples, one can mention the problems of optic damaging by a partially incoherent laser, stimulated Brillouin scattering in lens, and stimulated Raman scattering in crystals [6]. The IHS model is characterized by the following three steps.

(i) One neglects multiple amplifications in successive hot spots. This approximation is valid provided that the interaction length is smaller than (or comparable to) the correlation length of the laser field. Such a situation can be encountered in, e.g., inhomogeneous plasmas in which the resonance length for a given wave triplet is smaller than (or comparable to) the hot-spot length. In this limit, multiple amplifications in successive hot spots can be neglected due to the fact that the light backscattered in a given hot spot is out of resonance in any other hot spot it encounters on its way out of the plasma.

(ii) One approximates each hot spot near its maximum by a *given,* i.e., nonstochastic, intensity profile depending on the correlation function of the laser field and being the same for each hot spot [7,8]. This approximation follows from a result of the theory of homogeneous Gaussian fields stating that in the neighborhood of a high maximum at $\mathbf{x} = 0$, the random field looks, in probability, like $uC(\mathbf{x})$ with $O(1)$ random fluctuations, where u is the amplitude of the maximum and $C(\mathbf{x})$ is the correlation function of the field [7].

(iii) Finally, one averages over the hot-spot intensity to obtain the overall (macroscopic) reflectivity.

Although the IHS model has been extensively used since the seminal work by Rose and DuBois [5,8], the validity of these approximations has never been addressed, e.g., by comparing the IHS model results to those of the underlying convective amplifier. In this Letter, we undertake such a comparison in the simplest case of a one-dimensional (1D) linear convective amplifier. In this case, the most important approximation which makes the IHS model much more tractable than the stochastic convective amplifier is point (ii).

The starting point of this work is the following reasoning: in the linear case, the divergence of the average reflectivity, which determines the critical intensity, is due to the occurrence of hot spots with arbitrarily large $|u|$. For these hot spots, the $O(1)$ random fluctuations are negligible compared to the deterministic part $uC(\mathbf{x})$ and the approximation (ii) seems to be perfectly correct. Thus, in the case where the interaction length is smaller than the correlation length of the laser field [in order for (i) to be fulfilled], one expects the IHS model to give the *exact* value of the critical intensity. We show in the following that this is not the case: the IHS model properly gives the critical intensity and asymptotic (divergent) behavior of the amplification in the limit $L \ll l_c$ only, where L is the interaction length and l_c is the (longitudinal) correlation length of the laser field. However, for a reasonable form of the correlation function $C(\mathbf{x})$, the error made as $L \sim l_c$ is found to remain small so that the critical intensity given by the IHS model for $L \lesssim l_c$ is still in a good qualitative agreement with the exact value. The discrepancy observed for a small but finite L/l_c can be attributed to the failure of the approximation (ii), i.e., to the random fluctuations of the hot-spot field around the dominant part $uC(\mathbf{x})$.

We consider the following 1D stochastic convective amplifier:

$$\partial_x E(x) = g S^2(x) E(x), \tag{1}$$

where g is a (real) coupling constant playing the role of the average laser intensity, and S is a real homogeneous Gaussian field defined by $\langle S(x)\rangle = 0$, $\langle S(x)S(y)\rangle = C(x-y)$, and $C(0) = 1$. Expanding S according to the Karhunen-Loève expansion [7], one can express the moments of E in terms of the eigenvalues of the correlation function C. Namely, one finds that the amplification factor for $\langle E\rangle$, $A \equiv \langle E(L/2)\rangle / E(-L/2)$, is given by

$$A \equiv \langle e^{g\int_{-L/2}^{L/2} S^2(x)\,dx}\rangle = \prod_{n=1}^{+\infty} \frac{1}{\sqrt{1-2g\kappa_n}}, \tag{2}$$

where $\kappa_1 > \kappa_2 \cdots > \kappa_n > \ldots$ are the solutions to the eigenvalue equation $\int_{-L/2}^{L/2} C(x-y)\varphi_n(y)\,dy = \kappa_n \varphi_n(x)$, and φ_n are the corresponding orthonormal eigenfunctions. The critical coupling for $\langle E\rangle$ is thus given by $g_{cr} = 1/(2\kappa_1)$, and one has the asymptotic behavior

$$A \sim \frac{Z}{\sqrt{1-2g\kappa_1}} \qquad \left(g \uparrow \frac{1}{2\kappa_1}\right), \tag{3}$$

with $Z = \prod_{n=2}^{+\infty}(1-\kappa_n/\kappa_1)^{-1/2}$. A similar expression for $\langle E^N\rangle$ is obtained in which g is replaced by Ng.

As explained previously, one expects the IHS model to be able to retrieve this behavior *exactly* in the case $L \lesssim l_c$. The IHS model counterpart of Eq. (2) reads

$$A_{\mathrm{IHS}} = \frac{1}{\eta}\int_{I_{\min}}^{+\infty}\int_{-\eta/2}^{\eta/2} P(I)\mathcal{A}(I,x_0)\,dx_0\,dI, \tag{4}$$

with

$$\mathcal{A}(I,x_0) = e^{gI\int_{-L/2}^{L/2} C(x-x_0)^2\,dx}. \tag{5}$$

Here x_0 is the position of the hot spot giving rise to the amplification in the interaction region. Since S is homogeneous, x_0 is uniformly distributed over the "influence zone" $-\eta/2 \le x_0 \le \eta/2$ where $\eta \ge L$ is at most of the order of the correlation length. The quantity $P(I)$ denotes the average number of hot spots of intensity between I and $I + dI$ (with $I \equiv u^2$), within the influence zone. For $I \ge I_{\min}$, with $I_{\min} \simeq 6$ typically [9], it is given by [7]

$$P(I) = \frac{\eta|C''(0)|^{1/2}}{2\pi} e^{-I/2}. \tag{6}$$

The quantity η will not play any role in the following and needs not be further specified. Performing the x_0 integration by the steepest descent method, one obtains after some straightforward algebra

$$A_{\mathrm{IHS}} \sim \frac{Z_{\mathrm{IHS}}}{\sqrt{1-2g\kappa_{\mathrm{IHS}}}} \qquad \left(g \uparrow \frac{1}{2\kappa_{\mathrm{IHS}}}\right), \tag{7}$$

with $Z_{\mathrm{IHS}} = \{|C''(0)|\kappa_{\mathrm{IHS}}/[2C(L/2)|C'(L/2)|]\}^{1/2}$ and $\kappa_{\mathrm{IHS}} = \int_{-L/2}^{L/2} C(x)^2\,dx$. Note that Eq. (7) does not depend on η and $I_{\min}$, and that the x_0 integration, which is usually omitted in the applications of the IHS model, is necessary to get the correct exponent of the divergent factor. The quantities κ_{IHS} and Z_{IHS} are not equal to κ_1 and Z in general, except in the very restrictive limit $L/l_c \ll 1$ where the expressions of Z_{IHS} and κ_{IHS} coincide with κ_1 and Z at first and third order in L/l_c, respectively [cf. Eqs. (9) and (10)].

Figure 1 shows κ_1/L and κ_{IHS}/L for two simple forms of the correlation function: (a) $C(x) = [1 + (2x/l_c)^2]^{-1}$ and (b) $C(x) = \sin(3.79x/l_c)/(3.79x/l_c)$, where l_c is defined as the full width at half maximum (FWHM) of $C(x)$. It can be seen that the relative error on κ_1 remains small in the whole domain $L \lesssim l_c$. At $L = l_c$ one has $\Delta\kappa/\kappa \simeq 0.1$ in case (a) and $\Delta\kappa/\kappa \simeq 0.02$ in case (b), with $\Delta\kappa/\kappa \equiv 2|\kappa_1 - \kappa_{\mathrm{IHS}}|/(\kappa_1 + \kappa_{\mathrm{IHS}})$. We have checked that this result holds for a wide range of

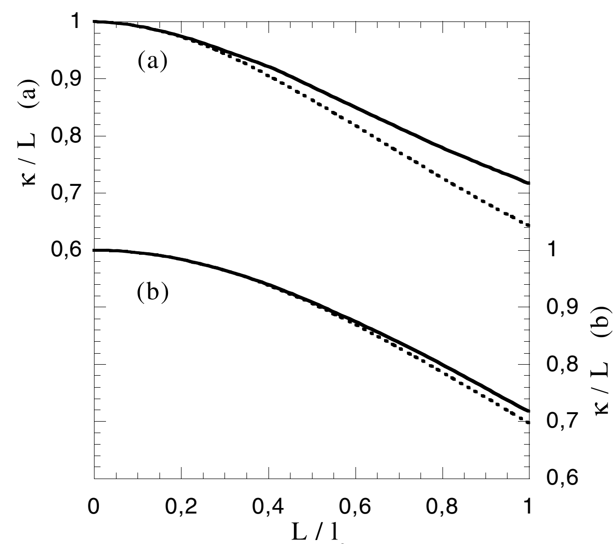

FIG. 1. κ_1/L (solid line) and κ_{IHS}/L (dotted line) as a function of L/l_c for two simple forms of the pump-field correlation function: (a) $C(x) = [1 + (2x/l_c)^2]^{-1}$ and (b) $C(x) = \sin(3.79x/l_c)/(3.79x/l_c)$.

reasonable forms of the correlation function. Hence, in the case $L \lesssim l_c$, the IHS model can be regarded as a good heuristic model giving a satisfactory estimate of the critical coupling. However, one should not overlook the inability of this model to yield the *exact* values of κ_1 and Z. This failure means that, in the limit $g \uparrow g_{cr}$, one cannot *a priori* justify the approximation (ii), and consequently the model itself, just on behalf of the fact that the statistically relevant hot spots have an arbitrarily large intensity. Instead, the validity of the critical coupling given by the IHS model depends on the correlation function $C(x)$ and should be discussed for each particular pump field $S(x)$.

Figure 2 shows κ_1/L and κ_{IHS}/L for a more complicated correlation function with two characteristic lengths: $C(x) = (1/2)\{[1 + (2x/l_1)^2]^{-1} + [1 + (2x/l_2)^2]^{-1}\}$. The correlation length, again defined as the FWHM of $C(x)$, is now $l_c = (l_1 l_2)^{1/2}$ and we have taken $l_2 = 100 l_1$. In this case one has $\Delta\kappa/\kappa \simeq 30\%$ for $L \simeq 0.3 l_c$ and $\Delta\kappa/\kappa \simeq 50\%$ at $L = l_c$, which makes the use of the IHS model for this particular correlation function very questionable. The origin of the discrepancy is related to the characteristic distance between neighboring hot spots, l_{HS}, which can be estimated heuristically from Eq. (6). This equation suggests $l_{\mathrm{HS}}^{-1} \simeq (2\pi)^{-1}|C''(0)|^{1/2} \int_{I_{\min}}^{+\infty} e^{-I/2}\, dI$, which gives $l_{\mathrm{HS}} \simeq (\pi l_1/2) e^{I_{\min}/2}$. Taking $I_{\min} = 6$, one has $I_{\min} > \log(l_2/l_1) = 4.61$ and $l_{\mathrm{HS}} > (l_1 l_2)^{1/2} = l_c$. It follows that, for $L \leq l_c$, there is at most one hot spot in the interaction region and the discrepancy must again be mainly attributed to the failure of (ii). This example of two-scale correlation function cannot be regarded as very representative of an *actual* laser field correlation function. Nevertheless, it is interesting to notice that in the case of two crossed laser beams with $O(1)$ crossing angle, the hot-spot intensity profile along one of the beams does have such a two-scale structure with, in addition, a short wavelength interference pattern superimposed on the central part of the hot spot. Although such an interference pattern seems to invalidate the use of the simple (first order) convective amplifier model (1), the result displayed in Fig. 2 suggests that one should be cautious in applying the IHS model to beam crossing experiments with a large crossing angle.

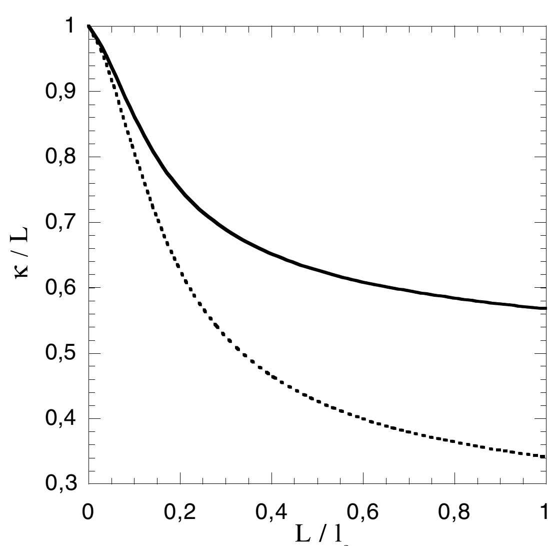

FIG. 2. κ_1/L (solid line) and κ_{IHS}/L (dotted line) as a function of L/l_c for a multiscale pump-field correlation function: $C(x) = (1/2)\{[1 + (2x/l_1)^2]^{-1} + [1 + (2x/l_2)^2]^{-1}\}$. The correlation length is $l_c = (l_1 l_2)^{1/2}$ and $l_2 = 100 l_1$.

As suggested by the previous results, for a small but finite L/l_c (or L/l_{HS} for a multiscale correlation), the discrepancy between the IHS model and the underlying stochastic amplifier comes from the random fluctuations of the hot-spot field around its deterministic component $uC(x)$. To prove this statement one must reconsider the IHS model without assuming the approximation (ii); i.e., one must replace the deterministic component $uC(x)$ by the full (stochastic) conditional field $S_u(x)$ corresponding to the realizations of S with a local maximum of height u at $x = 0$. In the following we show that the effects of the random fluctuations of S_u do cancel the deviations $\kappa_{\mathrm{IHS}} - \kappa_1$ and $Z_{\mathrm{IHS}} - Z$ at lowest order. In the limit of a small interaction region, $L \ll l_c$, the quantities κ_1 and Z can be obtained as power series of $\varepsilon \equiv (L/l_c)^2 \ll 1$. Taking L as unit length, expanding $C(x)$ as

$$C(x) = 1 - \varepsilon \frac{\Lambda}{2} x^2 + \varepsilon^2 \frac{M}{4!} x^4 + O(\varepsilon^3), \qquad (8)$$

where $\Lambda > 0$ and $M > \Lambda^2$, and determining the eigenvalues κ_n perturbatively in ε, one finds

$$\kappa_1 = 1 - \varepsilon \frac{\Lambda}{12} + \varepsilon^2 \frac{\Lambda^2 + 2M}{6!} + O(\varepsilon^3), \qquad (9a)$$

$$Z = 1 + \varepsilon \frac{\Lambda}{24} + O(\varepsilon^2). \qquad (9b)$$

On the other hand, from the expressions of κ_{IHS} and Z_{IHS} and the expansion (8), one obtains

$$\kappa_{\mathrm{IHS}} = 1 - \varepsilon \frac{\Lambda}{12} + \varepsilon^2 \left(\frac{3}{2}\right)^2 \frac{\Lambda^2 + M/3}{6!} + O(\varepsilon^3), \qquad (10a)$$

$$Z_{\mathrm{IHS}} = 1 + \varepsilon \frac{M}{48\Lambda} + O(\varepsilon^2). \qquad (10b)$$

According to Theorem 6.7.1 of Ref. [7], the conditional field S_u can be written as $S_u(x) = uC(x) + w_u b(x) + Y(x)$, where Y is a nonhomogeneous, zero-mean, Gaussian field, w_u is a random variable independent of Y, and $b(x) = (\varepsilon x^2/2)\sqrt{M - \Lambda^2} + O(\varepsilon^2)$. If one does not assume the approximation (ii), then the amplification factor $\mathcal{A}(I, x_0)$ on the right-hand side of Eq. (4) must be replaced by

$$\mathcal{B}(u, x_0) = \langle e^{g \int_{-1/2}^{1/2} S_u(x - x_0)^2\, dx} \rangle_{w_u, Y}, \qquad (11)$$

where $\langle \cdot \rangle_{w_u, Y}$ denotes the statistical average over the realizations of w_u and Y. The density for w_u is given by [7]

$$P(w_u) = \frac{\varepsilon(u\Lambda + w_u\sqrt{M - \Lambda^2})}{C(u)}\,\Theta(w_u)e^{-w_u^2/2}, \quad (12)$$

where $\Theta(w_u) = H(u\Lambda + w_u\sqrt{M - \Lambda^2})$, H is the Heaviside step function, and $C(u) \sim \sqrt{2\pi}\,u\varepsilon\Lambda$ as $u \to +\infty$. The corrections yielded by Y are found to be of higher order than $\kappa_1 - \kappa_{\text{IHS}}$ and $Z - Z_{\text{IHS}}$; we will thus take $Y = 0$ in the following. In this limit, the stochastic fluctuations of S_u reduce to a mere random variation of the curvature of the intensity profile near the hot-spot maximum. Performing the w_u integration by a steepest descent method, one finds that $\mathcal{B}(u, x_0)$ reduces to $\beta(I, x_0)\mathcal{A}(I, x_0)$ with $I \equiv u^2$ and where the function $\beta(I, x_0)$ is given by

$$\beta(I, x_0) = \left[1 - \varepsilon g\left(\frac{M}{\Lambda} - \Lambda\right)\left(\frac{1}{12} + x_0^2\right) + O(\varepsilon^2)\right]\exp\left[\frac{\varepsilon^2 g^2 I}{2}(M - \Lambda^2)\left(\frac{1}{12} + x_0^2\right)^2 + O(\varepsilon^3)\right]. \quad (13)$$

Averaging over w_u has promoted the term of S_u^2 proportional to uw_u, which was *a priori* negligible, to the status of a significant term, proportional to u^2, renormalizing the deterministic gain factor $gu^2\int_{-1/2}^{1/2} C(x - x_0)^2\,dx$ of the IHS model. Performing then the remaining integrations over x_0 and I in the limit $g \uparrow g_{cr}$, one recovers the correct asymptotic behavior (3), with κ_1 and Z given by Eqs. (9a) and (9b), respectively. The deviations $\kappa_{\text{IHS}} - \kappa_1$ and $Z_{\text{IHS}} - Z$ have been canceled, at lowest order, by the contribution of $\beta(I, x_0)$, i.e., by the effects of the random fluctuations of the hot-spot field around its deterministic component.

In this Letter, we have compared the IHS model results to those of the underlying stochastic amplifier. We have considered the case of a 1D linear amplifier in the limit where there is at most one hot spot per interaction length. We have checked that, for a wide class of pump-field correlation functions, the IHS model gives a satisfactory estimate of the exact critical coupling. We have then pointed out that this result is by no means a general result. Random fluctuations of the hot-spot field around its dominant, deterministic, component can spoil this qualitative agreement, even at the vicinity of the critical coupling where the IHS model was expected to be very accurate. It follows that the validity of the critical coupling given by the IHS model depends on the correlation function of the pump field and should be discussed in each particular case. We have given an example of a multiscale pump-field correlation function for which the discrepancy is found to be significant. Finally, for a small but finite L/l_c, we have *explicitly* shown that the discrepancy between the IHS model and the underlying stochastic amplifier must be attributed to the random fluctuations of the hot-spot field around its deterministic component.

The author thanks D. Pesme for interest and valuable discussions. He also thanks J. L. Lebowitz, H. A. Rose, P. Dai Pra, and A. Asselah for many fruitful discussions on related topics.

[1] G. Laval *et al.,* Phys. Fluids **20**, 2049 (1977); E. A. Williams, J. R. Albriton, and M. N. Rosenbluth, Phys. Fluids **22**, 139 (1979), and references therein.
[2] Y. Kato *et al.,* Phys. Rev. Lett. **53**, 1057 (1984).
[3] Ph. Mounaix, Phys. Plasmas **2**, 1804 (1995); Phys. Rev. E **52**, 1306 (1995).
[4] A. Asselah, P. Dai Pra, J. L. Lebowitz, and Ph. Mounaix, physics/0011010.
[5] H. A. Rose and D. F. DuBois, Phys. Rev. Lett. **72**, 2883 (1994).
[6] J. Garnier and C. Gouedard (private communication).
[7] J. Adler, *The Geometry of Random Fields* (Wiley, New York, 1981).
[8] H. A. Rose and D. F. DuBois, Phys. Fluids B **5**, 590 (1993).
[9] For a complex Gaussian field $P(I)$ behaves as $\exp(-I)$ instead of $\exp(-I/2)$ and $I_{\min} \simeq 3$.