Exact Phase Diagram for an Asymmetric Avalanche Process

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The Bethe ansatz method and an iterative procedure based on detailed balance are used to obtain exact results for an asymmetric avalanche process on a ring. The average velocity of particle flow, v, is derived as a function of the toppling probabilities and the density of particles, ρ. As ρ increases, the system shows a transition from intermittent to continuous flow, and v diverges at a critical point ρ_c with exponent α. The exact phase diagram of the transition is obtained and α is found to depend on the toppling rules.

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Avalanche dynamics is commonly found in extremal systems with movable elements near a border of stability. Relaxation of unstable states leads to the dispersive transport of particles involved into avalanches producing the typical long-tailed distribution of avalanche sizes [1]. Many physical phenomena such as interface depinning [2] or earthquakes [3] can be recast in terms of avalanches [4]. However, the most direct example is given by a class of granular systems, which exhibit intermittent avalanches explicitly and evolve naturally to a critical state [5]. One of the simplest lattice versions of granular systems is the "Oslo" model [6]. This is a critical slope model on a one-dimensional lattice with open boundaries. Grains are dropped at the left boundary with some driving rate and move along the lattice to the right every time the local slope of density exceeds some critical value. The critical slopes are dynamical random variables. This model has been shown to represent a large class of avalanche phenomena and undergoes a dynamical transition from intermittent to continuous flow when the driving rate increases [7]. Despite its apparent simplicity, the Oslo model resists analytical treatment, in contrast with deterministic sandpile models [8].

In this Letter, we introduce a solvable stochastic model of the asymmetric avalanche process (ASAP) on a ring. We use the Bethe ansatz method and an iterative procedure based on detailed balance to calculate the average velocity of particle flow, v, as a function of the toppling probabilities and density of particles, ρ. We find that as ρ increases at fixed toppling probabilities, the system shows a transition from intermittent to continuous flow, and v diverges at a critical point ρ_c with exponent α. We obtain the exact phase diagram of the transition and find that α depends on the toppling rules.

We define the ASAP as follows: In a stable state, P particles are located on a ring of N sites. During the infinitesimal time interval dt, each particle has a probability dt of jumping one step to the right. If any site x contains a number of particles $n(x) > 1$, it becomes unstable, and must relax immediately by spilling to its right either n particles with probability μ_n or $n - 1$ particles with probability $1 - \mu_n$. The relaxation stops when all sites become stable again with $n(x) \leq 1$. This random process is a generalization of a model studied by Maslov and Zhang (MZ) [9]. We let μ_n depend on n and use periodic boundary conditions [10,11]. To induce an avalanche dynamics in a closed system, we use infinitesimally slow directed Poissonian driving of particles through the lattice instead of adding particles from outside [7,12,13]. The parameters that control the transition to continuous flow in our case are the density of particles and the toppling probabilities unlike the driving rate in the Oslo model. In the MZ model as well as our model, the critical height and random toppling rules can be considered as corresponding to the random critical slope in the Oslo model. MZ have shown that their model gives avalanche dynamics with a critical distribution of avalanche sizes [9].

It is instructive to compare our model with the asymmetric exclusion process (ASEP), which has a long history in the literature [14–18]. The usual presentation of the ASEP is given by a master equation for the probability $P_t(x_1, \ldots, x_P)$ of finding P particles at time t on sites $x_1, \ldots, x_P$ of a ring of N sites. During any time interval dt, each particle jumps with probability dt to its right if the target site is empty. This elementary restriction leads to a nontrivial problem of evaluation of the steady state properties, which can be solved by the Bethe ansatz. The main difference between the ASEP and ASAP lies in the degree of reorganization of a configuration of particles during the time interval dt. In the ASEP, a new configuration either differs from the initial one by a position of a single particle, if the motion is allowed, or remains unchanged if the motion is forbidden. In the ASAP, the motion of particles is always possible and the new configuration may be completely different from the previous one if an avalanche occurs. Therefore, the Bethe ansatz method, to be applied to the ASAP, should be essentially extended.

Before handling the Bethe ansatz, we give an intuitive derivation of the quantity of interest. The velocity of the particle flow is defined as the average number of steps of all particles involved in an avalanche. For large N, the basic features of the system can be already seen in a simple description under the assumption that the occupation numbers of all sites are uncorrelated and equal to the density $\rho = P/N$. Let $P_x(n)$ be the expected number of the events when n particles are spilled from a site $x-1$ to x during an avalanche. They satisfy the Markov equation

$$P_{x+1}(n) = \sum_m P_x(m) w_{m,n} \tag{1}$$

with the transition probabilities

$$\begin{aligned} w_{n-1,n} &= \rho \mu_n , \\ w_{n,n} &= \rho(1-\mu_{n+1}) + (1-\rho)\mu_n , \\ w_{n+1,n} &= (1-\rho)(1-\mu_{n+1}) . \end{aligned}$$

Here the transition probability $w_{m,n}$ corresponds to the event where m particles flow into site x from site $x-1$ and then n particles flow out from site x to site $x+1$. The transition probabilities satisfy the normalization $\sum_n w_{m,n} = 1$.

Then we consider the total number of spillings during an avalanche $P(n) = \sum_{x=1}^{N} P_x(n)$ irrespectively of sites where they occur. One can verify that $P(n)$ corresponds to a stationary solution of the same Markov equation (1). The solution can easily be found recursively from the detailed balance condition

$$P(n) w_{n,n+1} = P(n+1) w_{n+1,n} . \tag{2}$$

The first term of the recursion, $P(1)$, can be derived from the expected numbers of stops: $P(1)(1-\rho) = 1$. This condition implies that an avalanche stops every time when the single particle hits an empty site. Resolving the recursion, one obtains

$$P(n) = \frac{1}{\rho}\left(\frac{\rho}{1-\rho}\right)^n \prod_{k=2}^{n} \frac{\mu_k}{1-\mu_k}, \qquad n \geq 2 . \tag{3}$$

Then, the velocity of flow is

$$v = \sum_{n=1}^{\infty} n P(n) . \tag{4}$$

Close to the critical point ρ_c the average velocity of particle flow is expected to have a power law divergency $(\rho_c - \rho)^{-\alpha}$ [10,11]. The exponent α is not universal and depends on toppling rules. In the Oslo model [6], the divergence of the average avalanche size is exponential when the driving rate approaches its critical value [7]. The formula (3) allows one to construct a class of models parametrized by an arbitrary exponent α which controls the divergence of the velocity of the flow in the vicinity of the critical point. To this end, consider probabilities μ_n in the form

$$\frac{\mu_n}{1-\mu_n} = \frac{2\mu}{1-\mu} \frac{n-2+\alpha}{n\alpha} , \tag{5}$$

where μ is a parameter and $\mu \in [0,1)$. For all $\alpha \in (0,\infty]$ and all $n \geq 2$ we have $\mu_n \in [0,1]$. Now, from the general result Eq. (4), we obtain

$$v = \frac{1}{1-\rho}\left(1 - \frac{2\mu\rho}{\alpha(1-\mu)(1-\rho)}\right)^{-\alpha} . \tag{6}$$

This expression reveals power law singularity with an arbitrary exponent α. The critical density ρ_c as a function of parameter μ is plotted in Fig. 1. Solid lines correspond to different values of α. Below these lines, the velocity is finite and avalanches are intermittent. The area above the lines corresponds to the phase of continuous flow. The broken line in Fig. 1 corresponds to the particular choice of set $\{\mu_n\}$

$$\mu_n = \mu \frac{1-(-\mu)^{n-1}}{1+\mu} . \tag{7}$$

This set of parameters ensures complete integrability of the ASAP. Below, we shall see that for this set $\{\mu_n\}$ formula (3) coincides with that obtained in the thermodynamic limit from the exact Bethe ansatz solution. At the same time, the simplified solution (6) does not imply that the problem can be solved for finite N and P. It is one of the goals of the Letter, to show that the choice of μ_n in the form (7) ensures the exact Bethe ansatz solution of the asymmetric avalanche process.

The Bethe ansatz approach is free from assumption about noncorrelated occupancy of lattice sites and deals directly with the probability $P_t(x_1,\ldots,x_P)$ of finding P particles on sites $x_1,\ldots,x_P$. When the distances between neighboring particles are greater than 1, the master equation for $P_t(x_1,\ldots,x_P)$ describes free directed diffusion:

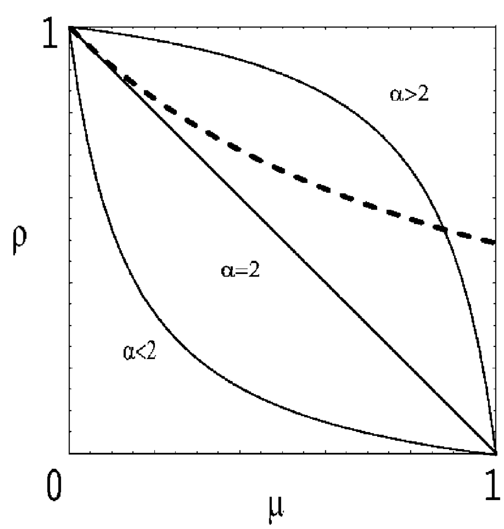

FIG. 1. The phase diagrams for different toppling rules. The broken line separates the intermittent flow phase from the continuous flow phase for the case $\mu_n = \mu(1-\mu_{n-1})$. Solid lines represent phase boundaries for models parametrized by α for $\alpha = 1/4$, $\alpha = 2$, and $\alpha = 16$.

$$\partial_t P_t(x_1,\ldots,x_P) = \sum_k [e^{\gamma} P_t(\ldots,x_k - 1,\ldots) - P_t(\ldots,x_k,\ldots)]. \quad (8)$$

We introduce the auxiliary parameter γ, which allows one to use the largest eigenvalue of the kinetic equation as the generating function of the distance Y_t traveled by a particle during the time t [16]. The limit $\gamma \to 0$ will finally be taken, so that the solution $P_t(x_1,\ldots,x_P)$ of Eq. (8) will become the probability.

The absence of neighboring particles ensures that only the probabilities of stable configuration enter Eq. (8). In order to make this equation valid even when any two particles are neighbors, one has to exclude the "unphysical" terms that appear in the right-hand side with the help of the boundary conditions

$$P_t(\ldots,x,x,\ldots) = (1-\mu)e^{\gamma}P_t(\ldots,x-1,x,\ldots) + \mu e^{2\gamma}P_t(\ldots,x-1,x-1,\ldots). \quad (9)$$

We consider this relation as recursion. Here the unphysical term $P_t(\ldots,x,x,\ldots)$ is expressed through another unphysical term $P_t(\ldots,x-1,x-1,\ldots)$ and so on, until we get a final expression in terms of stable configurations only. In other words, the diffusion equation (8) describes the slow Poisson dynamics of particles, while the recursion relation gives the fast avalanche process.

Similar boundary conditions can be written for the case of three or more neighboring particles. However, applicability of the Bethe ansatz requires all of them to be reduced to the two-particle boundary condition (9). Thus, the ansatz can be used only if the probabilities μ_n for spilling all n particles from a site can be represented recursively as a cascade of two-particle topplings, each spilling either two particles with probability μ or one particle with probability $(1-\mu)$. In this way, we obtain the recursion relation

$$\mu_2 = \mu, \qquad \mu_n = \mu(1-\mu_{n-1}) \quad (10)$$

that gives the expression (7) for toppling probabilities μ_n.

Now we can use the Bethe ansatz for the eigenfunction of Eq. (8):

$$P_t(x_1,\ldots,x_P) = e^{N\Lambda t}\sum_{\sigma_{(1,\ldots,P)}} A(z_{\sigma_1},\ldots,z_{\sigma_P}) \times z_{\sigma_1}^{-x_1},\ldots,z_{\sigma_P}^{-x_P},$$

where the sum is over all permutations $\sigma_{(1,\ldots,P)}$ of the set of indexes $1,2,\ldots,P$. The eigenvalue is

$$\Lambda = \frac{e^{\gamma}}{N}\sum_{i=1}^{P} z_i - \frac{P}{N}. \quad (11)$$

The boundary condition (9) fixes the relation of amplitudes in the Bethe ansatz

$$\frac{A(\ldots,z_i,z_j,\ldots)}{A(\ldots,z_j,z_i,\ldots)} = -\frac{1-(1-\mu)e^{\gamma}z_i - \mu e^{2\gamma}z_i z_j}{1-(1-\mu)e^{\gamma}z_j - \mu e^{2\gamma}z_i z_j}.$$

If we also impose periodic boundary conditions, we obtain the Bethe equations

$$z_k^{-N} = (-1)^{P-1}\prod_{j=1}^{P}\frac{1-(1-\mu)e^{\gamma}z_j - \mu e^{2\gamma}z_j z_k}{1-(1-\mu)e^{\gamma}z_k - \mu e^{2\gamma}z_j z_k}. \quad (12)$$

For a finite N, the eigenvalue $\Lambda = 0$ corresponds to the solution $z_j \to 1$ as $\gamma \to 0$ for all $j = 1,\ldots,P$. In the thermodynamic limit, the dependence $z_j(\gamma)$ becomes discontinuous at $\gamma = 0$. For any $\gamma > 0$, even infinitesimally small, the roots z_j form a finite-size contour around the trivial solution $z_j = 1, j = 1,\ldots,P$.

Let us introduce variables x by

$$z = e^{-\gamma}\frac{1-x}{1+\mu x} \quad (13)$$

and assume that the solutions $\{x_j\}$ are distributed along a contour Γ which is symmetric with respect to complex conjugation. Then, we introduce the density of roots $R(x)$ along the contour such that

$$\rho = \frac{1}{2\pi i}\int_{\Gamma} R(x)\,dx. \quad (14)$$

In the thermodynamic limit, we can write Eq. (11) in the integral form

$$\Lambda = -\frac{1+\mu}{2\pi i}\int_{\Gamma}\frac{xR(x)}{1+\mu x}\,dx. \quad (15)$$

Taking the logarithm of Eq. (12), we obtain the Bethe equation

$$\gamma = -p(x) + \frac{1}{2\pi i}\int_{\Gamma}\Theta(x/y)R(y)\,dy + 2\pi i h(x), \quad (16)$$

where

$$p(x) = \ln\left(\frac{1+\mu x}{1-x}\right), \qquad \Theta(x) = \ln\left(\frac{1+\mu x}{x+\mu}\right).$$

The function $h(x)$ is defined so that $2\pi i dh/dx = R(x)$ and $h(x_0) = -h(\bar{x}_0) = \rho/2$, where x_0 and $\bar{x}_0$ are the endpoints of Γ. When $\gamma = +0$, the integration contour is closed, and the values $x_0 = \bar{x}_0 = -|x_0|$ are real and negative. In this case, the solution of the integral equation (16) can be found easily:

$$R_0(x) = \frac{\rho}{x} - \frac{1}{x-1}. \quad (17)$$

When $\gamma > 0$, the contour becomes disconnected, which corresponds to the so-called conical point at the phase diagram of the asymmetric six-vertex model [19]. In the vicinity of $\gamma = 0$, Eq. (16) can be treated perturbatively [20]. Namely, we can look for the solution $R(x)$ as an expansion in powers of ϵ which is a small deviation of the argument of x_0 from π so that $x_0 = |x_0|e^{-i(\pi-\epsilon)}$

$$R(x) = R_0(x) + \epsilon R_1(x) + \epsilon^2 R_2(x) + \cdots . \qquad (18)$$

The zero order term is given by (17). The calculation of the first order shows that the perturbative solution exists only if $R_0(x_0) = 0$, which fixes the absolute value of x_0 relating it to the density of particles $\rho = x_0/(x_0 - 1)$. Next orders can be obtained from expressions found by Bukman and Shore [20].

$$R_1(x) = 0, \qquad (19a)$$

$$R_2(x) = \frac{1}{6}\frac{1 + x_0}{(1 - x_0)^3}\frac{x_0}{x}, \qquad (19b)$$

$$R_3(x) = \frac{1}{3\pi(1 - x_0)^2}\sum_{n\neq 0}\frac{n(-\mu)^{|n|}}{1 - (-\mu)^{|n|}}\left(\frac{x_0}{x}\right)^{n+1}. \qquad (19c)$$

If we are looking for γ also in the form of the perturbative expansion in ϵ, the first nonvanishing term is of third order

$$\gamma = -\epsilon^3\frac{x_0}{3\pi(1 - x_0)^2} + O(\epsilon^5). \qquad (20)$$

The eigenvalue Λ up to the same order is

$$\Lambda(\gamma) = \epsilon^3\frac{(\mu + 1)x_0}{3\pi(1 - x_0)^2} \times \left(\frac{x_0}{(1 + x_0\mu)^2} - \sum_{s=1}^{\infty}\frac{s(-\mu)^{2s-1}x_0^s}{1 - (-\mu)^s}\right). \qquad (21)$$

Considering Λ as the generating function of the numbers of steps involved into an avalanche, we can write the average velocity as

$$v = \frac{1}{\rho}\frac{\partial\Lambda}{\partial\gamma}\bigg|_{\gamma=+0} = \frac{1}{\rho}\frac{\partial\Lambda}{\partial\epsilon}\bigg/\frac{\partial\gamma}{\partial\epsilon}. \qquad (22)$$

From (20) and (21), using (7) we get finally

$$v = \frac{(1 - \rho)(1 + \mu)}{[1 - \rho(1 + \mu)]^2} + \frac{1}{\rho}\sum_{n=1}^{\infty}\frac{n(-1)^n\mu^{2n}}{\mu_{n+1}}\left(\frac{\rho}{1 - \rho}\right)^n. \qquad (23)$$

It is easy to see that this result is equivalent to Eqs. (3) and (4) if μ_n is chosen in the form (7).

The velocity of flow given by Eq. (23) diverges with critical exponent $\alpha = 2$ at $\rho = \rho_c = 1/(1 + \mu)$. For large avalanches, $n \gg 1$, the probability μ_n tends to the constant $\mu_\infty = \mu/(1 + \mu)$ and hence arguments by Maslov and Zhang [9] can be used for finding $P(s)$. At the critical line, $P(s) \sim 1/s^\tau$ where $\tau = 4/3$ is the universal exponent for a broad class of directed models [21]. Derivation of this exponent needs the only fact that the number n of spilled particles at every step of the avalanche makes fair random walk when n is large. It would be interesting to find the condition of changing the universality class as, for example, the condition of multiple topplings in a two-dimensional directed sandpile model [22,23].

One can see that the exact expression of the average velocity in the thermodynamic limit can be obtained both with the simplified arguments and with the Bethe ansatz. However if an average value, e.g., diffusion coefficient, contains higher cumulants of the distance Y_t traveled by a particle, the Bethe ansatz remains the only appropriate tool. The Bethe ansatz approach allows one also to take into account effects of the finite size of the system.

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