

Shear Viscosity of Strongly Coupled $\mathcal{N} = 4$ Supersymmetric Yang-Mills Plasma

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Using the anti-de Sitter/conformal field theory correspondence, we relate the shear viscosity η of the finite-temperature $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in the large N , strong-coupling regime with the absorption cross section of low-energy gravitons by a near-extremal black three-brane. We show that in the limit of zero frequency this cross section coincides with the area of the horizon. From this result we find $\eta = \frac{\pi}{8} N^2 T^3$. We conjecture that for finite 't Hooft coupling $g_{\text{YM}}^2 N$ the shear viscosity is $\eta = f(g_{\text{YM}}^2 N) N^2 T^3$, where $f(x)$ is a monotonic function that decreases from $\mathcal{O}(x^{-2} \ln^{-1}(1/x))$ at small x to $\pi/8$ when $x \rightarrow \infty$.

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Introduction.—At finite temperatures, the large distance, long time behavior of gauge theories is described, as in any other fluid, by a hydrodynamic theory [1]. To write down the hydrodynamic equations one has to know the thermodynamics (i.e., the equation of state) of the medium, as well as the transport coefficients: the shear and the bulk viscosities, the electrical conductivity [in the presence of a U(1) gauge group], and the diffusion constants (in the presence of conserved global charges). Knowledge of these quantities in hot gauge theories is crucial for numerous applications, the most notable of which belong to the physics of the electroweak phase transition in the early Universe [2] and of the quark-gluon plasma possibly created in heavy-ion collisions [3].

When the gauge coupling is small (which requires, in the case of QCD, temperatures much larger than the confinement scale), both the equation of state and the transport coefficients are calculable perturbatively. At strong coupling (i.e., at temperatures not much larger than the confinement scale), thermodynamics can be found nonperturbatively by lattice simulations, but transport coefficients are beyond the reach of all current numerical techniques. This situation is very unfortunate, since the quark-gluon plasma one hopes to create in heavy-ion experiments has relatively low temperature at which the perturbation theory works very poorly.

Lacking methods to reliably compute the transport coefficients of finite-temperature QCD, one should try to gain insight into models where these coefficients can be computed nonperturbatively. Recently, powerful techniques based on the anti-de Sitter/conformal field theory (AdS/CFT) correspondence have been developed, establishing, in particular, the connection between the $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in the large coupling, large N limit and classical ten-dimensional gravity on the background of black three-branes [4–7]. This allows one to perform analytical calculations in a strongly coupled four-dimensional gauge theory.

In this Letter, we compute the shear viscosity η of the strongly coupled finite-temperature $\mathcal{N} = 4$ SYM theory (the bulk viscosity of this theory vanishes due to scale invariance). We first relate, using previously known results from the AdS/CFT correspondence, the shear viscosity with the absorption cross section of low-energy gravitons falling perpendicularly onto near-extremal black three-branes. We further show that this cross section is equal to the area of the horizon, in a way very similar to the case of black holes [8]. These facts provide enough information for us to find that $\eta = \frac{\pi}{8} N^2 T^3$, provided both the 't Hooft coupling and N are large. Remarkably, the shear viscosity approaches a constant value in the large 't Hooft coupling limit, and this value is related to the area of the horizon of the black brane.

The viscosity.—To start our discussion, we briefly review the notion of viscosity in the context of finite-temperature field theory. Consider a plasma slightly out of equilibrium, so that there is local thermal equilibrium everywhere but the temperature and the mean velocity slowly vary in space. We define, at any point, the local rest frame as the one where the three-momentum density vanishes: $T_{0i} = 0$. The stress tensor, in this frame, is given by the constitutive relation

$$T_{ij} = \delta_{ij} p - \eta \left(\partial_i u_j + \partial_j u_i - \frac{2}{3} \delta_{ij} \partial_k u_k \right) - \zeta \delta_{ij} \partial_k u_k, \quad (1)$$

where u_i is the flow velocity, p is the pressure, and η and ζ are, by definition, the shear and bulk viscosities, respectively. In conformal field theories such as the $\mathcal{N} = 4$ SYM theory, the energy momentum tensor is traceless, $T^\mu{}_\mu = 0$, so $\varepsilon \equiv T_{00} = 3p$ and the bulk viscosity vanishes identically, $\zeta = 0$.

All kinetic coefficients can be expressed, through Kubo relations, as the correlation functions of the corresponding currents [9]. For the shear viscosity, the correlator is that

of the stress tensor,

$$\begin{aligned}\eta &= \lim_{\omega \rightarrow 0} \frac{1}{2\omega} \int dt d\mathbf{x} e^{i\omega t} \langle [T_{xy}(t, \mathbf{x}), T_{xy}(0, 0)] \rangle \\ &= \lim_{\omega \rightarrow 0} \frac{1}{2\omega i} [G_A(\omega) - G_R(\omega)],\end{aligned}\quad (2)$$

where the average $\langle \dots \rangle$ is taken in the equilibrium thermal ensemble, and G_A and G_R are the advanced and retarded Green functions of T_{xy} , respectively. In Eq. (2), the Green functions are computed at zero spatial momentum. Though Eq. (2) can, in principle, be used to compute the viscosity in weakly coupled field theories, this direct method is usually very cumbersome, since it requires resummation of an infinite series of Feynman graphs. This calculation has been explicitly carried out only for scalar theories [10]. A more practical method is to use the kinetic Boltzmann equation, which gives the same results as the diagrammatic approach [11].

For gauge theories at weak coupling, $g^2 N \ll 1$, where throughout this paper $g \equiv g_{\text{YM}}$ is the gauge coupling, the shear viscosity has the following parametric behavior,

$$\eta = C \frac{N^2 T^3}{(g^2 N)^2 \ln(1/g^2 N)}. \quad (3)$$

Basically, η is proportional to the product of the energy density $\varepsilon \sim N^2 T^4$ and the transport mean free time $\tau \sim [(g^2 N)^2 \ln(1/g^2 N) T]^{-1}$. The numerical coefficient C in Eq. (3), in principle, can be computed by solving the linearized Boltzmann equation [12].

The relation to graviton absorption.—The key observation underlying this work is that the right hand side of the Kubo formula (2) is known to be proportional to the classical absorption cross section of gravitons by black three-branes [13,14]. For completeness, we recall here the basic argument leading to this correspondence. Consider, in type IIB string theory, a configuration of N D3-branes stacked on top of each other. The low-energy theory living on the branes is the $\mathcal{N} = 4$ U(N) SYM theory. On the other hand, if N is large, the stack of D3-branes has large tension, which curves space-time. In the limit of large 't Hooft coupling $g^2 N$, the three-brane geometry has small curvature and can be described by supergravity. Therefore, we have two descriptions of the same physics in terms of strongly coupled gauge theory on the branes and classical gravity on a certain background.

If one sends a graviton to the brane, there is some probability that it will be absorbed. On the gravity side, the absorption cross section can be calculated by solving the wave equation on the background metric. On the gauge theory side, the rate of graviton absorption measures the imaginary part of the stress tensor–stress tensor correlator, since gravitons polarized parallel to the brane are coupled to the stress-energy tensor of the degrees of freedom on the brane. The relation between the absorption cross section $\sigma(\omega)$ of a graviton with energy ω , polarized parallel

to the brane (say, along the xy directions) and falling at a right angle on the brane is related to the correlator in field theory as [13,14]

$$\sigma(\omega) = \frac{\kappa^2}{\omega} \int dt d\mathbf{x} e^{i\omega t} \langle [T_{xy}(t, \mathbf{x}), T_{xy}(0, 0)] \rangle, \quad (4)$$

where $\kappa = \sqrt{8\pi G}$, G being the ten-dimensional gravitational constant. The relation (4) has been explicitly verified for zero-temperature field theory (or extremal black branes, in the gravity language) [13,14]. Such a check is possible because there is a nonrenormalization theorem for the correlator of the stress-energy tensor [15].

At finite temperature T , Eq. (4) relates the graviton absorption cross section by a near-extremal black brane having the Hawking temperature equal to T with a correlator in the hot SYM theory [16]. Since there is no supersymmetry and no nonrenormalization theorem is known to work at finite temperature, one cannot explicitly verify the relation (4). We instead view Eq. (4) as a prediction of the theory. In particular, taking the $\omega \rightarrow 0$ limit one can relate $\sigma(\omega = 0)$ with the shear viscosity of hot SYM plasma,

$$\eta = \frac{1}{2\kappa^2} \sigma(0). \quad (5)$$

Equation (5) implies that, for nonextremal black branes, the graviton absorption cross section must not vanish in the limit of zero frequency (in contrast to the extremal case where $\sigma(\omega) \sim \omega^3$ at small ω [13,14]), and, by computing the zero-frequency value of σ one obtains the shear viscosity of the hot SYM plasma. The problem of computing the shear viscosity is now reduced to a problem of classical gravity.

The metric of a nonextremal black three-brane has the form [17,18]

$$\begin{aligned}ds^2 &= H^{-1/2}(r) [-f(r)dt^2 + d\mathbf{x}^2] \\ &\quad + H^{1/2}(r) [f^{-1}(r)dr^2 + r^2 d\Omega_5^2],\end{aligned}\quad (6)$$

where $H(r) = 1 + R^4/r^4$ and $f(r) = 1 - r_0^4/r^4$. The extremal case corresponds to $r_0 = 0$; the limit relevant for us is the near-extremal one, $r_0 \ll R$. This metric has a horizon at $r = r_0$. From the existence of this horizon one should expect $\sigma(0)$ to be nonvanishing. Running ahead, we will show, by solving the wave equation on the metric (6), that $\sigma(0)$ is *equal* to the area of the horizon,

$$\sigma(0) = \pi^3 r_0^3 R^2 \quad (7)$$

(the numerical coefficient π^3 is simply the area of the unit five-sphere). Using the formula for the Hawking temperature of the metric (6),

$$T = \frac{r_0}{\pi R^2}, \quad (8)$$

and the relation between R and N which is obtained by identifying the Arnowitt-Deser-Misner mass per unit volume of the three-brane with the tension of a stack of N

D3-branes [13],

$$R^4 = \frac{\kappa N}{2\pi^{5/2}}, \quad (9)$$

we find the shear viscosity to be

$$\eta = \frac{\pi}{8} N^2 T^3. \quad (10)$$

This is the main result of the paper. Up to a constant, the shear viscosity is equal to the entropy density [19,20]. Both quantities are proportional to the area of the horizon.

Solution to the radial wave equation.—Now let us show how Eq. (7) is obtained. We have to solve the s -wave radial equation for a minimally coupled scalar (such as the graviton polarized parallel to the brane), $\partial_\mu(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = 0$. In the metric (6) this equation acquires the form

$$\phi'' + \frac{5r^4 - r_0^4}{r(r^4 - r_0^4)} \phi' + \omega^2 \frac{r^4(r^4 + R^4)}{(r^4 - r_0^4)^2} \phi = 0. \quad (11)$$

The method we use to solve Eq. (11) is the same matching method that was used in the extremal case [13,14]. Since ultimately we are interested in the limit $\omega \rightarrow 0$, we will assume $\omega \ll T$. More details about our method, as well as the solution to the radial equation in the opposite limit $\omega \gg T$ and for higher partial waves, can be found in Ref. [21]. Earlier attempts to compute the absorption rate by nonextremal black branes were made in Refs. [22]. As in the extremal case, we search for the solution in several regions and match the result wherever the regions overlap. Let us go from small r to large r , starting from the horizon $r = r_0$. The first region is the one just outside the horizon: $r > r_0$, $r - r_0 \ll r_0$. In this case Eq. (11) has the form

$$\phi'' + \frac{\phi'}{r - r_0} + \frac{\lambda^2}{16} \frac{\phi}{(r - r_0)^2} = 0, \quad (12)$$

where $\lambda = \omega/(\pi T) \ll 1$. The solution to Eq. (12) is

$$\phi = A \left(1 - \frac{r_0}{r}\right)^{-i\lambda/4}, \quad (13)$$

where we have chosen the sign of the exponent so that the solution corresponds to an incoming wave at the horizon. When λ is small, (13) is basically a constant, A , except for an exponentially small region near r_0 . In the next region, $r_0 < r \ll \omega^{-1}$ (excluding r exponentially closed to r_0), the term proportional to ω^2 in the left-hand side of Eq. (11) can be dropped. Indeed,

$$\omega^2 r^8 (r^4 - r_0^4)^{-2} \ll (r - r_0)^{-2} \quad (14)$$

due to $r \ll \omega^{-1}$, and

$$\omega^2 r^4 R^4 (r^4 - r_0^4)^{-2} \ll (r - r_0)^{-2} \quad (15)$$

since $\omega \ll T \sim r_0 R^{-2}$. Equation (11) now has the form

$$\phi'' + \frac{5r^4 - r_0^4}{r(r^4 - r_0^4)} \phi' = 0, \quad (16)$$

which possesses a trivial solution,

$$\phi = A, \quad (17)$$

which matches smoothly with Eq. (13). Finally, in the outermost region, $r \gg R \gg r_0$, Eq. (11) is simplified to

$$\frac{d^2 \phi}{dr^2} + \frac{5}{r} \frac{d\phi}{dr} + \omega^2 \phi = 0, \quad (18)$$

which can be solved in terms of the Bessel functions,

$$\phi(r) = \alpha \frac{J_2(\omega r)}{(\omega r)^2} + \beta \frac{Y_2(\omega r)}{(\omega r)^2}, \quad r \gg R. \quad (19)$$

The regimes of validity of Eq. (17), $r_0 < r \ll \omega^{-1}$, and of Eq. (19), $r \gg R$, has an overlap since $\omega^{-1} \gg R$ (this is the consequence of $R \gg r_0$ and $\omega \ll T$). In order for Eq. (19) to match with Eq. (17) in the overlapping region, one should require

$$\alpha = 8A, \quad \beta = 0. \quad (20)$$

The field at large distances can be decomposed into an incoming wave and an outgoing wave,

$$\phi(r) = 4A \left[\frac{H_2^{(1)}(\omega r)}{(\omega r)^2} + \frac{H_2^{(2)}(\omega r)}{(\omega r)^2} \right]. \quad (21)$$

The absorption probability P is the ratio of the flux at $r = r_0$ from Eq. (13) and the flux from the incoming wave in Eq. (21). We find

$$P = \frac{\pi}{32} \omega^5 r_0^3 R^2. \quad (22)$$

Since the absorption cross section σ is related to P by [8]

$$\sigma = \frac{32\pi^2}{\omega^5} P, \quad (23)$$

we arrive to Eq. (7), which coincides with the area of the horizon. This is very similar to the universal result for black holes [8].

Notice that in deriving Eq. (7) we require ω to be much smaller than the Hawking temperature. The absorption cross section will deviate substantially from the zero-frequency limit if ω is of the order of T . In particular, the next correction to Eq. (7) is of the order of ω^2/T^2 with a computable coefficient [21].

Discussion.—We have shown that the shear viscosity can be computed in the strongly coupled $\mathcal{N} = 4$ SYM theory from the AdS/CFT correspondence. Now let us try to interpret the result (10). The power of T in η is completely fixed by the dimensionality of η and the scale invariance of the theory. The factor N^2 apparently comes from the number of degrees of freedom in the plasma. It is remarkable that the shear viscosity approaches a constant

value as one sends the 't Hooft coupling to infinity. From the relation $\eta \sim \varepsilon\tau$, one can interpret this behavior as the indication that the “relaxation time” τ remains of order T^{-1} (but not much smaller) in the strong coupling limit. Since the inverse relaxation time is comparable to the energy per degree of freedom, the strongly coupled plasma cannot be viewed as a collection of particles, and the formula $\eta \sim \varepsilon\tau$ does not apply in the strict sense. However, one should expect that the counting of the powers of N still works. This counting, combined with the expressions for η in the weak-coupling [Eq. (3)] and strong-coupling [Eq. (10)] limits, suggests that for finite 't Hooft coupling g^2N the shear viscosity has the form

$$\eta = f(g^2N)N^2T^3, \quad (24)$$

where $f(x) \sim x^{-2} \ln^{-1}(1/x)$ when $x \ll 1$ and $f(x) = \pi/8$ when $x \gg 1$. It is most likely that $f(x)$ is a monotonic function of x . One way to verify this conjecture is to compute the $\mathcal{O}(1/g^2N)$ correction to η in the strong coupling limit. If $f(x)$ is monotonic, then this correction must be positive. This is analogous to the behavior of the free energy [20], except that at small coupling $\eta \rightarrow \infty$, while the free energy remains finite.

Recalling that $\sigma(\omega)$ deviates substantially from $\sigma(0)$ when $\omega \sim T$, we see that the hydrodynamic theory can describe processes occurring during times much larger than T^{-1} , but breaks down for those whose typical time scale is of order or less than T^{-1} . One also should expect hydrodynamics to work at spatial distances much larger than T^{-1} , but not at distances of order or less than T^{-1} . This is consistent with T^{-1} playing the role of the relaxation time in the limit $g^2N \rightarrow \infty$. There is apparently no separation of scales in the strong coupling regime that would make a kinetic description possible: T^{-1} is the only time/length scale. Thus, the viscosity η cannot be computed from a Boltzmann-type equation.

In this paper, we have confined our attention to the most important transport coefficient—the shear viscosity. As mentioned above, the bulk viscosity vanishes identically due to the exact scale invariance of the $\mathcal{N} = 4$ SYM theory. It would be useful to compute other transport coefficients in this theory (for example, the diffusion constant of the R charges) at strong coupling using the AdS/CFT correspondence.

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