Virtual Quantum Subsystems

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(Received 7 March 2001; published 25 July 2001)

The physical resources available to access and manipulate the degrees of freedom of a quantum system define the set $\mathcal{A}$ of operationally relevant observables. The algebraic structure of $\mathcal{A}$ selects a preferred tensor product structure, i.e., a partition into subsystems. The notion of compoundness for quantum systems is accordingly relativized. Universal control over virtual subsystems can be achieved by using quantum noncommutative holonomies

DOI: 10.1103/PhysRevLett.87.077901 PACS numbers: 03.67.Lx, 03.65.Fd, 03.65.Ud

In the past few years we witnessed a strong revival of interest about the notion of quantum *entanglement* [1]. This is mainly due to the essential role that such a concept is supposed to play in quantum information processing (QIP) [2]. Whenever one has a compounded (or multipartite) quantum system, in the space of admissible states there exist states which display uniquely quantum correlations. These states are referred to as *entangled* and correspond algebraically to the existence, in a vector space obtained by a tensor product, of vectors $|\psi\rangle$ that are *not* expressible by a simple product, e.g., $|\psi_1\rangle \otimes |\psi_2\rangle$.

Given a physical system S, the way to subdivide it into subsystems is in general by no means unique. However, it is a widespread practice in theoretical physics as well as in everyday life to consider different partitions in subsystems with the dependence of both the physical regime and the necessities of the description. It is indeed a quite common experience to refer sometimes to a system, e.g., an atom, as elementary and sometimes as a composite, e.g., made out of electrons and nucleons. The *emergence* of a distinguished multipartite structure is strongly dependent on the physical regime, e.g., the energy scale, at which one is working and on the set of observations (experiments) the observer is interested in. This is of course a well-known lesson from the history of physics, e.g., fundamental vs composite particles, weak-strong coupling dualities, renormalization group, etc.

Clearly even the notion of entanglement is affected by some ambiguity being *relative* to the selected multipartite structure. States that are entangled with respect to a given partition in subsystems can be separable with respect to each other. Conversely, states of a system S that is regarded as elementary can be viewed as entangled once S is endowed with a multipartite structure. In this case, one is in the, somehow paradoxical, situation of having entanglement seemingly *without* entanglement.

The above ambiguity is removed as soon as, according to some criterion, a preferred multipartite structure is selected among the family of all possible partitions into subsystems. This selection has in most cases a well-defined meaning: the system S is viewed as composed by $S_1, S_2, \ldots$ if one has some operational access (is able to "access," "control," "measure") to the individual degrees of freedom of $S_1, S_2, \ldots \gtrsim$. In other terms it is the set of "available" interactions that individuates the relevant multiparty decomposition and not an *a priori*, God-given partition into elementary subsystems. In this Letter we shall make an attempt to formalize the ideas brought about by these simple remarks. Our final goal is to provide a satisfactory algebraic definition of what a quantum subsystem is in an operationally motivated framework.

Let us stress that the notion of a *virtual* subsystem that we shall introduce admits as a particular instance the one of quantum code [3,4] and noiseless subsystems [5,6]. This remark should make clear that virtual subsystems *already* play an important role in QIP. In particular, error-avoiding quantum codes, i.e., decoherence-free [4], have also been recently experimentally observed [7,8].

Compoundness and tensor products.—Let us begin by recalling the basic algebraic structures associated with compoundness. Let S_1 and S_2 be two *classical* systems with configuration manifolds M_i $(i = 1, 2)$. Roughly speaking the associated quantum systems have state spaces given by $\mathcal{H}_i = \mathcal{F}(M_i)$, where $\mathcal{F}$ denotes some suitable (complex-valued) function space over the M_i's, e.g., L^2-summable functions. Notice that these spaces (actually *Abelian* C^*-algebras [9]) are the classical "observable" spaces; the quantum spaces are given by the operator (non-Abelian) algebras $\mathrm{End}(\mathcal{H}_i)$. In the classical realm the manifold associated with the joint systems $S_1 \vee S_2$ is given by the *Cartesian* product $M_1 \times M_2$. It follows that, at the *quantum* level, one has $\mathcal{H}_{S_1 \vee S_2} = \mathcal{H}_1 \otimes \mathcal{H}_2$; indeed $\mathcal{F}(M_1 \times M_2)$ is given by a suitable closure of $\mathcal{F}(M_1) \otimes \mathcal{F}(M_2)$. This basic functorial identity is the *algebraic* ground for the quantum theory axiom associating a bipartite system with a state space given by the tensor products of the state spaces describing the *subsystems*. The extension to N-partite systems is obvious. One has another elementary, yet remarkable, functorial relation given by the *canonical* isomorphism $\mathrm{End}(\mathcal{H}_1 \otimes \mathcal{H}_2) \cong \mathrm{End}(\mathcal{H}_1) \otimes \mathrm{End}(\mathcal{H}_2)$. Even in the quantum realm the observable algebra associated with a joint system is given by the tensor product of the subsystem subalgebras.

Our key observation is that different types of compoundness can emerge in the same system when one considers different sets of observables as the physical ones. Indeed quite often it makes sense to refer to a subalgebra $\mathcal{A}$ (rather than the full operator algebra) as the *physical* observable algebra. Limitations of physical resources may lead one to select a specific class of operators to be considered as realizable. For instance, energy supply limitations lead naturally to restrictions to operators X which have vanishing matrix elements between energy eigenstates whose energy difference exceeds some bound E. At the dynamical-algebraic level the selection of a particular multiparty decomposition means that the algebra of (operationally relevant) observables $\mathcal{A}$ has a tensor product structure (TPS), i.e., $\mathcal{A} \cong \otimes \mathcal{A}_i$ such that all the observables belonging to the individual $\mathcal{A}_i$'s can be effectively implemented.

Before proceeding to general constructions it is useful to consider a very simple example in which one has a set of subsystems (degrees of freedom) associated with a rapidly growing sequence of energy scales. Starting from the ground state and increasing the energy available, one is able to excite more and more subsystems which, at lower energy, were frozen. This situation is realized, for instance, in systems in which one has confined directions or in the cases in which an adiabatic decoupling between fast and slow degrees of freedom has been performed: the effective dimensionality of the system is a function of the energy scale.

The TPS manifold.—Let us consider a Hilbert space $\mathcal{H} \cong \mathbb{C}^n$ with *a priori no* tensor product structure. A first very natural question is *how many nonequivalent TPS's can be assigned over* $\mathcal{H}$? More physically, in how many different ways can $\mathcal{H}$ be viewed as the state space of a multipartite quantum system? If n is a prime number there are no possibilities: the system is *elementary*. If n is *not* prime it has a nontrivial prime factorization: $n = \prod_{i=1}^r p_i^{n_i}$ $(p_i < p_{i+1})$. If the exponent n_i of the ith prime factor of n is not one, then several regroupings are possible, e.g., $r = 1, p_1 = 2, n_1 = 3 \Rightarrow 3 = 1 + 1 + 1, 3 = 1 + 2$, corresponding to the state-space factorizations $\mathbb{C}^8 \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^8 \cong \mathbb{C}^2 \otimes \mathbb{C}^4$. When more than one p_i appear in the decomposition of n we see that many other possibilities of writing n as a product of integers arise. In general, given n, we introduce the set of factorizations $\mathcal{P}_n = \{P \subset \mathbf{N}/\prod_{m\in P} m = n\}$, where $\mathbf{N}$ denotes the set of natural numbers.

Given that a factorization $P = \{n_1 \leq n_2 \leq \ldots \leq n_{|P|}\} \in \mathcal{P}_n$ of n is assigned, one has the (noncanonical) isomorphisms φ: $\mathcal{H} \mapsto \otimes_{j=1}^{|P|}\mathbb{C}^{n_j}$. In the following such isomorphisms will be referred to as TPS's over $\mathcal{H}$, and subsystems of the associated multiparty decomposition will be referred to as virtual.

Given a distinguished TPS, say φ_0, one can identify the group of unitaries $\mathcal{U}(\mathcal{H})$ and $\mathcal{U}(\otimes_j\mathbb{C}^{n_j})$ via the algebra isomorphism $U \mapsto \varphi_0^{-1} \circ U \circ \varphi_0$. A suitable quotient of this latter unitary group parametrizes the space of nonequivalent TPS's. Indeed two elements, U and W, of $\mathcal{U}(\otimes_j\mathbb{C}^{n_j})$ define equivalent TPS's if either $U = U_1WU_2$, where the U_i's are *multilocal* transformations, i.e., $U_i \in \prod_{k=1}^{|P|} U(n_k)$ $(i = 1, 2)$, or the U_i's are *permutations* of factors with equal dimension. In the first case the TPS's differ just by a change of the basis in each factor, and in the second by the order of the factors that in turn amounts simply to a relabeling of the subsystems. The space of *nonequivalent* TPS's over $\mathbb{C}^n$ will be denoted by $\mathcal{T}_n$.

Once a given multiplicative partition (n_i) of n is chosen along with a particular φ, one has $\mathcal{H} = \otimes_{i=1}^N\mathcal{H}_i$ $[\mathcal{H}_i := \varphi(\mathbb{C}^{n_i})]$ and then $\mathrm{End}(\mathcal{H}) \cong \otimes_{i=1}^N \mathcal{A}_i$, where $\mathcal{A}_i := \mathrm{End}(\mathcal{H}_i)$. For any set of unitaries in $\mathcal{H}$ labeled by the elements λ of some manifold $\mathcal{M}$, e.g., external fields, one can define $\mathcal{A}_i(\lambda) := U_\lambda\mathcal{A}_iU_\lambda^\dagger$ $(i = 1, \ldots, N)$ that describes a family of multipartite structures over $\mathcal{H}$ parametrized by points of $\mathcal{M}$. As noticed above not all the points of $\mathcal{M}$ necessarily correspond to different TPS's. Indeed it can happen that different λ's can result in the same structure, e.g., $U_\lambda\mathcal{A}_iU_\lambda^\dagger = \mathcal{A}_i$. If a state is entangled (product) with respect to a TPS labeled $\lambda \in \mathcal{T}_n$ it will be referred to as λ entangled (λ product).

If E: $\mathcal{H} \mapsto \mathbb{R}_0^+$ denotes an entanglement measure over $\mathcal{H}$ with respect to a given TPS, say $\lambda = 0$, one has that $E_\lambda := E \circ U_\lambda$ is a λ-entanglement measure. In turn the latter provides a natural measure of the "distance" between the TPS at $\lambda \neq 0$ and that at $\lambda = 0$. Indeed it appears quite natural to say that the more the $\lambda = 0$ product states are λ-entangled the more the TPS at λ differs from the one at the origin. To make this idea quantitative, one has to make it independent of the particular state; this can be done either by maximizing or by taking the average over all the 0-product states. In this latter case, one finds that the distance one is looking for is nothing but the (square root of) *entangling power* of U_λ [10]: $e(U) = \int d\psi_1\, d\psi_2\, E(U|\psi_1\rangle \otimes |\psi\rangle_2)$. Here the integral is done with respect to the uniform, e.g., Haar, measure over the pure product state manifold.

In order to exemplify the notion of the TPS manifold we now introduce a family of TPS's over an infinite-dimensional state space parametrized by a group of $N \times N$ matrices. Let us consider N harmonic oscillators. The global state space is given by $\mathcal{H}_N := \otimes_{i=1}^N\mathcal{H}_i$, where each of the factors is the single boson Fock space, i.e., $\mathcal{H}_i = \mathrm{span}\{|n\rangle\}_{n\in\mathbf{N}}$ associated with the annihilation and creation operators a_i and $a_i^\dagger$ $(a_i^\dagger a_i|n\rangle = n|n\rangle)$. Let $U \in U(N)$ be a complex $N \times N$ unitary matrix. The operators $a_i^U := \sum_{j=1}^N U_{ji}a_j$ $(i = 1, \ldots, N)$ represents new bosonic *modes*, i.e., $[a_i^U, a_j^{U\dagger}] = \delta_{ij}, [a_i^U, a_j^U] = 0$; moreover one has $\mathcal{H} = \otimes_{j=1}^N\mathcal{H}_i^U$, where the $\mathcal{H}_j^U$'s are the Fock spaces associated with the a_j^U's. Notice that the Fock vacuum $|0\rangle := \otimes_i|0\rangle_i$ is U independent, i.e., $a_J^U|0\rangle = 0 (\forall\, U, j)$. One has $\mathcal{H}_j^U \cong \mathcal{A}_j^U|0\rangle$, where $\mathcal{A}_j^U$

is the algebra generated by a_j^U and $a_j^{U\dagger}$. States such as $a_j^{U\dagger}|0\rangle$ are disentangled with respect to the TPS defined by the given U but entangled with respect to the one associated with, e.g., $U = \mathbb{1}$.

Virtual bipartitions.—Now we address the following issue: When is it legitimate to consider a pair of observable algebras as describing a bipartite quantum system? Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are two *commuting* $*$-subalgebras of $\mathcal{A} := \mathrm{End}(\mathcal{H})$ such that the subalgebra $\mathcal{A}_1 \vee \mathcal{A}_2$ they generate, i.e., the minimal $*$-subalgebra containing both $\mathcal{A}_1$ and $\mathcal{A}_2$, amounts to the whole $\mathcal{A}$, and moreover one has the (noncanonical) algebra isomorphism,

$$\mathcal{A}_1 \vee \mathcal{A}_2 \cong \mathcal{A}_1 \otimes \mathcal{A}_2 . \tag{1}$$

The standard, *genuinely* bipartite, situation is of course $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $\mathcal{A}_1 = \mathrm{End}(\mathcal{H}_1) \otimes \mathbb{1}$, and $\mathcal{A}_2 = \mathbb{1} \otimes \mathrm{End}(\mathcal{H}_2)$. If $\mathcal{A}_i' := \{X/[X, \mathcal{A}_1] = 0\}$ denotes the *commutant* of $\mathcal{A}_1$, in this case one has $\mathcal{A}_i' = \mathcal{A}_2$.

It is important to mention that a prototypical and ubiquitous situation described by Eq. (1) is when $\mathcal{A}_1$ and $\mathcal{A}_2$ are *local* observable algebras associated with disjoint regions of space at equal time. More generally such an independence of local degrees of freedom, e.g., quantum fields, is encoded in terms of commutativity between observables supported on causally disconnected domains [11]. Notice also that the spatial separation between parties, e.g., Alice and Bob, is a common assumption in protocols for quantum communication, e.g., teleportation [2].

The point of view advocated in this Letter is to consider condition (1) as the *definition* of a bipartite system, regardless of the "real" compoundness or not of the underlying state space. Accordingly we shall consider as a real entanglement the one occurring in that case. The (nearly obvious) point is that, in order to take computational advantage of this virtual entanglement, one must have *access* to, i.e., to be able to control, the subalgebras $\mathcal{A}_{1,2}$. As far as the operations in $\mathcal{A}_1$ and $\mathcal{A}_2$ being easily realizable (accessible) in the lab we shall consider them as primitive and local, regardless of how they look at the original level.

The theory of *noiseless subsystems* [5,6,12] provides an important exemplification as a well as a source of inspiration for the approach to compoundness advocated here. Let us consider a system made of N real subsystems, e.g., qubits. Suppose that the algebra of relevant interactions is given by $\mathcal{A}_1 \cup \mathcal{A}_1'$, where $\mathcal{A}_1$ [9] is

$$\mathcal{A}_1 \cong \oplus_J \mathbb{1}_{n_J} \otimes M_{d_J}(\mathbb{C}) . \tag{2}$$

This decomposition reads at the state-space level as $\mathcal{H} \cong \oplus_J \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J}$. For a fixed label J, one finds that the elements of $\mathcal{A}_1$ ($\mathcal{A}_1'$) act as the identity on the $\mathbb{C}^{n_J}$ ($\mathbb{C}^{d_J}$) factor. This means that the system is viewed, for all practical purposes, as a bipartite system, in which the observables of the first (second) subsystems are given by $\mathcal{A}_1$ ($\mathcal{A}_1'$) . For collective decoherence, $\mathcal{A}_1$ is the interaction algebra generated by couplings with the environment invariant under qubit permutations, while $\mathcal{A}_i'$ is given by any linear combination of permutation operators [6]. In particular the latter algebra is generated by exchange, i.e., Heisenberg-like operators between the different pairs of qubits [12,13].

Generally speaking Eq. (2) shows in which sense an observable algebra $\mathcal{A}_1$ ($\mathcal{A}_1'$) is associated with a collection of virtual subsystems, i.e., the $\mathbb{C}^{d_J}$ ($\mathbb{C}^{n_J}$) factors, labeled by its spectrum. It is worth observing that when $\mathcal{A}_1$ is *Abelian* all the d_J's are equal to 1. In this case, if $n_J > 1$, the Jth factor of the state-space decomposition describes a type of hybrid bipartite system in which one of the factors is quantum, whereas the other represents a classical system with a one-point configuration space. This is exactly the situation one meets in the case of quantum codes, both error correcting [3] and error avoiding [4]. In this latter case the algebra $\mathcal{A}_1$ is generated by the operators coupling the computing system to its environment and $\mathcal{A}_i'$ is the set of interactions necessary to perform computations entirely within the decoherence-free sector [6].

To make clear the connection with the quantum error correction, let us consider a set $\{X_i\}_{i=1}^k$ of $k \leq n$ linear independent traceless "parity" operators over $\mathcal{H} \cong (\mathbb{C}^2)^{\otimes n}$, such that $X_i = X_i^\dagger, X_i^2 = \mathbb{1}$, and $[X_i, X_j] = 0$ $(i, j = 1, \ldots, k)$. Following standard arguments of quantum error correction [3], one can show that the X_i's generate an Abelian algebra $\mathcal{A} \cong \mathbb{C}\mathbf{Z}_2^k$. The associated state-space decomposition is given by

$$\mathcal{H} \cong \oplus_{J \in \mathbf{Z}_2^k} \mathbb{C}^{2^{n-k}} \otimes \mathbb{C} \cong \mathbb{C}^{2^{n-k}} \otimes \mathbb{C}^{2^k}. \tag{3}$$

It is easy to see that the commutant of $\mathcal{A}$ contains the algebra of operators over the first factor in the decomposition above This means that the set of operators with well-defined parities defines and controls a virtual subsystem of n-k bits. Analogously the set of "odd" operators ($\{O/\exists\ i\{X_i, O\} = 0\}$) defines and controls the second k-qubit subsystem. For instance, the parity $X_1 := \sigma_x \otimes \mathbb{1}$ defines the natural bipartite structure over $(\mathbb{C}^2)^{\otimes 2}$, whereas $X_1' = \sigma_x^{\otimes 2}$ defines TPS such that states such as $2^{-1/2}(|00\rangle \pm |11\rangle)$ are disentangled. Notice that in error correction theory the first (second) subsystem is related to the code (syndrome). For any unitary U, the operators $X_i(U) := UX_iU^\dagger$ span an algebra isomorphic to $\mathcal{A}$ above. Again, one has a continuous set of TPS's parametrized by points of a unitary group [14].

Turning back to the characterization of pairs of (finite-dimensional) subalgebras satisfying Eq. (1) by using Eq. (2) it is easy to prove the following [15].

Proposition.—Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two commuting $*$-subalgebras of a finite-dimensional $*$-algebra $\mathcal{A}$. A necessary and sufficient condition for the validity of (1) is that $\mathcal{A}_1 \cap \mathcal{A}_1' = \mathbb{C}\mathbb{1}$, i.e., $\mathcal{A}_1$ is a factor.

Holonomic control on subsystems.—In this paragraph we show that the holonomic approach to quantum computation [16] provides a natural setting for the issue of information processing within a (virtual) subsystem.

Let $X \in \mathrm{End}(\mathcal{H}) \cong M_{nd}(\mathbb{C})$ be a Hermitian operator with a spectrum of d isodegenerate eigenvalues, i.e., $X = \sum_{i=1}^{d} x_i \sum_{k=1}^{n} |ki\rangle\langle ki|$, and $\{U_\lambda\}_{\lambda\in\mathcal{M}} \subset U(\mathcal{H})$ be a set of unitaries parametrized by the point of some (control) manifold $\mathcal{M}$. Then the set of $X(\lambda) := U_\lambda X U_\lambda^\dagger$ is a family that in the generic case, for sufficiently large $D = \dim\mathcal{M}$, satisfies the conditions for (universal) holonomic quantum computation [16] on the n-dimensional degenerate eigenspace $C_i = \mathrm{span}\{|ki\rangle\}_{k=1}^{n} \cong \mathbb{C}^n \otimes |i\rangle$ $(i = 1,\ldots,d)$ of $\mathbb{1}_n \otimes X$. This implies that the *holonomy group* $\mathrm{Hol}(A_i)$ associated with the connection $u(n)$-valued 1-forms, $A_i^{ab} = \langle a| \otimes \langle i|U_\lambda^\dagger dU_\lambda|b\rangle \otimes |i\rangle$, $d := \sum_{\mu=1}^{D} d\lambda_\mu\, \partial_\mu$ $(a, b = 1,\ldots,n)$, is the whole $U(C_i) \cong U(n) \otimes |i\rangle\langle i|$ [16]. By denoting collectively with A the set of the A_i's, one can therefore write that

$$\mathrm{Hol}(A) \cong \oplus_{i=1}^{d} U(n) \otimes |i\rangle\langle i| \supset U(n) \otimes \mathbb{1}_d\,. \tag{4}$$

The last inclusion tells us that in the generic case the holonomy group of A will contain the whole unitary group of the $\mathbb{C}^n$ subsystem. Once the holonomic family $\{X(\lambda)\}_\lambda$ is given, any transformation, i.e., computations in the first subsystem, can be generated holonomies. Notice that, since for the real quantum case one must have $n \geq 2$, the holonomy group is necessarily non-Abelian.

Conclusions.—We analyzed some of the consequences of the nonuniqueness of the decomposition of a given system S into subsystems. Such nonuniqueness implies, at the quantum level, a fundamental ambiguity about the very notion of entanglement that accordingly becomes a *relative* one. One can parametrize the space of all possible partitions, i.e., tensor product structures, of a n-dimensional quantum state space by the points of a set $\mathcal{T}_n$. The fact of considering all the points in $\mathcal{T}_n$ on the same footing (which amounts to establishing a *democracy* between different TPS's) provides a relativization of the notion of entanglement. Without further physical assumption, no partition has an ontologically superior status with respect to any other. The subsystems associated with all these possible, i.e., *potential*, multiparty decompositions were referred to as virtual. A distinguished point of $\mathcal{T}_n$ is selected, i.e., made *actual* only once the relevant algebra $\mathcal{A}$ of "physical" observables is given. Indeed considering a given partition as the privileged partition has a strong operational meaning, in that it depends on the set of resources effectively available to access and control the degrees of freedom of S. Different sets of resources give rise to different physically relevant partitions. We provided several examples of natural, though hidden, multipartite structures arising from the given algebraic structure of $\mathcal{A}$. We briefly showed that the holonomic approach to quantum computation provides one natural way to address the issue of controllability within virtual subsystems. We believe that this democratic approach to quantum compoundness is, on the one hand, sound from the conceptual point of view, and, on the other hand, possibly relevant to QIP.

The author acknowledges R. R. Zapatrin and M. Rasetti for stimulating discussions and encouragement.

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[14] Let us consider two qubits and the family by $U_\lambda := \exp(i\lambda S) = \cos\lambda \mathbb{1} + i \sin\lambda S$; where $S|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$. If $X_1 := \sigma_x \otimes \mathbb{1}$, one has $X_1(U_\lambda) = \cos^2\lambda X_1 + \sin^2\lambda \mathbb{1} \otimes \sigma_x + i/2 \sin 2\lambda [S, X_1]$. Clearly $X_1(\pi/2) = \mathbb{1} \otimes \sigma_x$: the TPS associated with $\lambda = \pi/2$ amounts to just the exchange between the two subsystems. All the points of $[0, \pi/2)$ correspond to nonequivalent TPS's.
[15] We can assume $\mathcal{A}_2 = \mathcal{A}_1'$. If $\mathcal{A}_1$ is a factor, one has $\mathcal{A}_1 \cong \mathbb{1}_n \otimes M_d(\mathbb{C})$ and then $\mathcal{A}_2 = M_n(\mathbb{C}) \otimes \mathbb{1}_n$. There follows that $\mathcal{A}_1 \vee \mathcal{A}_2 \cong M_n(\mathbb{C}) \otimes M_d(\mathbb{C}) = \mathcal{A}_1 \otimes \mathcal{A}_2$. When $\mathcal{A}_1$ is *not* a factor Eq. (2) implies $\mathcal{A}_2 \subset \oplus_J M_{n_J}(\mathbb{C}) \otimes \mathbb{1} d_J$, then $\mathcal{A}_1 \vee \mathcal{A}_2 = \oplus_J M_{n_J}(\mathbb{C}) \otimes M_{d_J}(\mathbb{C})$. A comparison of the dimension of this latter algebra with that of $\mathcal{A}_1 \otimes \mathcal{A}_2$ completes the proof.
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