Focusing of Noncircular Self-Similar Shock Waves

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We study the focusing of noncircular shock waves in a perfect gas. We construct an explicit self-similar solution by combining three convergent plane waves with regular shock reflections between them. We then show, with a numerical Riemann solver, that there are initial conditions with smooth shocks whose intermediate asymptotic stage is described by the exact solution. Unlike the focusing of circular shocks, our self-similar shocks have bounded energy density.

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The focusing of shock waves is a subject that has received a great deal of attention from the mathematical and physical communities [1,2] since the seminal work of Guderley [3]. In this problem, a convergent shock wave compresses a region occupied by undisturbed gas, concentrating energy in a small region of space. Here we discuss convergent shocks in ideal gases. This is an idealized model of inertial confinement experiments, where ionization effects and multiphase phenomena are neglected.

The flow is described by the conservation laws for the momentum, mass, and energy in a polytropic gas that evolves adiabatically:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p \,,$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \,, \tag{1}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\mathbf{v}] = 0 \,.$$

Here $\mathbf{v}$ is the velocity field, ρ is the gas density, p is the pressure, and $E = \rho \mathbf{v}^2/2 + p/(\gamma - 1)$ is the energy per unit volume of a gas with adiabatic constant γ [4].

For 2D and 3D radially symmetric flows, Guderley [3] found similarity solutions of the second kind [5], in which the velocity can be written as

$$\mathbf{v}(\mathbf{x}, t) = \frac{\mathbf{x}}{(T - t)} F\left(\frac{|\mathbf{x}|}{(T - t)^{\alpha}}\right), \tag{2}$$

where T is the focusing time, α is the similarity exponent which depends on γ, and F is the scaled velocity function [3,6,7]. Bilbao *et al.* accurately computed the similarity exponents α for a range of γ values in [8]. These self-similar solutions are expected to describe the flow as $t \to T$ in a small neighborhood of the focusing point $\mathbf{x} = 0$.

There are three relevant questions regarding similarity solutions.

(i) If axial symmetry is enforced, does a Guderley self-similar solution really describe the asymptotic focusing process as $t \to T$?

(ii) When $\gamma > 1.909\,19$ in 2D and when $\gamma > 1.869$ in 3D, the self-similar solution is not unique [6–8]. Indeed, there is a continuum of α values for each γ in the appropriate range. Which solution, if any, is selected by the dynamics?

(iii) Axially symmetric solutions are unstable with respect to symmetry-breaking perturbations [1,4]. What is the corresponding asymptotic behavior? Are there focusing self-similar solutions without circular symmetry?

Numerical simulations by Voloshinov [6] suggest that the answer to (i) is "yes," and that, when α is not uniquely determined by γ, the selected value of α is the one that yields the smoothest solution, as conjectured by Gel'fand. These computations are rather sketchy and further work needs to be done in this direction. We will return to this question on another occasion. Here we focus on question (iii).

Whitham used the *shock-dynamics approximation* [4] to derive an explicit approximate formula for the exponent α. In this approximation, a number of rays are defined as the orthogonal trajectories of the succesive portions of the shock, and then the propagation of the shocks between the rays is treated as the approximate 1D flow in a tube of solid walls. In this formulation, the evolution depends only on the shock shape and the local Mach number. The approximation is valid provided the flow behind the shock does not affect the shock itself.

The dynamics of the flow behind a convergent noncircular shock are very rich, as illustrated by the work of Demmig and Petersen [1]. Their numerical study of the full gas-dynamics problem, including the flow behind the shock, shows that the shock suffers fragmentation: The principal incident shock forms corners, and develops a pattern of reflected shocks behind it. Their results are in agreement with the experiments of Takayama and Watanabe [2] with perturbed cylindrical shocks.

Using the shock-dynamics approximation, Whitham also found that circular shocks are unstable with respect to small perturbations [4]. Later in collaboration with Schwendeman [9], he found approximate solutions for polygonal convergent shocks of k sides. These solutions show that Mach reflections are generated at the vertices of the polygonal shock, and that for $k \geq 3$ the vertices are periodically interchanged with the sides, producing an alternating shock wave. It is interesting to remark that the approximation used by Schwendeman and Whitham in [4] does not describe regular reflections, so these polygonal solutions are expected to be accurate only when the angles between shocks are large enough to produce Mach reflections. For triangular shocks, as we shall see (see Fig. 2 below), it is possible to have either Mach or regular reflections depending on the Mach number. We concentrate on the case of regular reflection.

In this Letter, we construct a 2D equilateral triangular exact self-similar solution which accounts for both the flow ahead and behind of the shocks, and show that appropriately perturbed circular shocks converge to this solution at focusing. This solution has a bounded density of energy accumulation, in contrast with the Guderley solution and the polygonal solutions of Schwendeman.

Let us consider three convergent shock waves as shown in Fig. 1A. The incident shocks I form an equilateral triangle whose sides propagate at velocity V towards its center. Inside the triangle, the unperturbed gas has zero velocity, constant density $\rho_1 > 0$, and pressure $p_1 > 0$. The sound velocity is $c_1 = \sqrt{\gamma p_1/\rho_1}$, and the Mach number of the incident shocks is $M = V/c_1$.

Suppose that at each corner of the triangle there is a *regular shock reflection* [12] (we will discuss this assumption later). The angle 2ϕ between the reflected shocks R must be computed along with the flow outside the triangle. Let us now consider a reference system where one of the corners is stationary (see Fig. 1B). This reference system moves to the left with velocity $V_x = -V/\sin\theta$ along the symmetry line S. In these coordinates, the steady oblique shock I refracts the horizontal flow incident on the left with velocity $v_x = V/\sin\theta$ and then this flow is refracted again by shock R. In this manner, we reduce the problem to a regular shock reflection problem, where the symmetry line replaces a reflecting solid wall. In our case there is an additional constraint on the reflected angle, $\phi \leq \pi/3$, which means that reflected shocks in Fig. 1A do not cross each other. Note that, since the converging shocks have a constant velocity V, the energy density remains bounded.

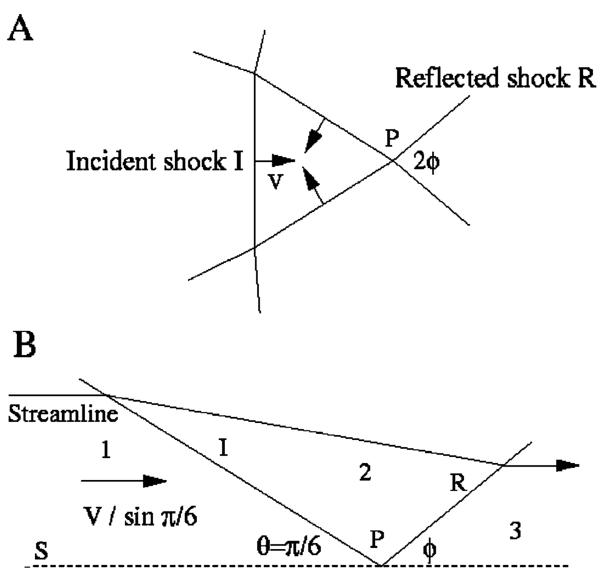

FIG. 1. (A) Sketch of three convergent shock waves. (B) On the system of reference of one of the corners, the problem is equivalent to a shock-reflection problem.

The regular reflection problem was solved by von Neumann [10,11]. He showed that the problem can be reduced to a quadratic equation. Here we write the solution in terms of the reflected angle ϕ in order to facilitate the formulation of the constraint $\phi \leq \pi/3$. The velocity field, density, and pressure in regions 2 and 3 (see Fig. 1B) are supposed to be constants. They can be computed using the oblique Rankine-Hugoniot conditions across the incident and reflected shocks. By requiring the outflow in region 3 to be parallel to the symmetry line S, the following quadratic equation for the reflected angle may be derived:

$$A\tan^2\phi + B\tan\phi + C = 0, \tag{3}$$

with

$$A = (M^2 - 1)[1 + \gamma + M^2(1 + 3\gamma + 2\gamma^2)],$$
$$B = -2\sqrt{3}[\gamma(2 + \gamma)M^4 + 2M^2 - 1],$$

and

$$C = M^4(4\gamma^2 - \gamma - 3) + M^2(-2\gamma^2 + 6\gamma + 4) - \gamma + 1.$$

This quadratic equation has two roots. One has to select the root corresponding to the smaller reflected angle, $\tan\phi = (-B - \sqrt{B^2 - 4AC})/2A$, because the other root yields sub-sonic velocities at region 3, making the reflection unstable [11]. The selected root gives the angle ϕ as shown in Fig. 1A. The resulting piecewise constant solution is self-similar, because the whole pattern scales linearly with time as the triangle shrinks. The constraint $\phi \leq \pi/3$ is satisfied for an equilateral triangular shock provided the roots of Eq. (3) are real, as can be verified by evaluating numerically the values of ϕ.

The stability condition for the self-similar solutions is related to the transition criteria from regular reflection to Mach reflection. A comprehensive review is given by Ben-Dor [12,13], and a brief summary with explicit transition formulas is published in [11]. There are basically two criteria for the regular-Mach transition: the so-called mechanical equilibrium criterion (in which the Mach stem has zero length and moves perpendicularly to the flow) and the detachment criterion [in which $B^2 - 4AC = 0$ so that the two roots on Eq. (3) are equal]. These criteria are depicted in Fig. 2. Shocks with an incidence angle θ for which the point (M, θ) lies above the upper curve in

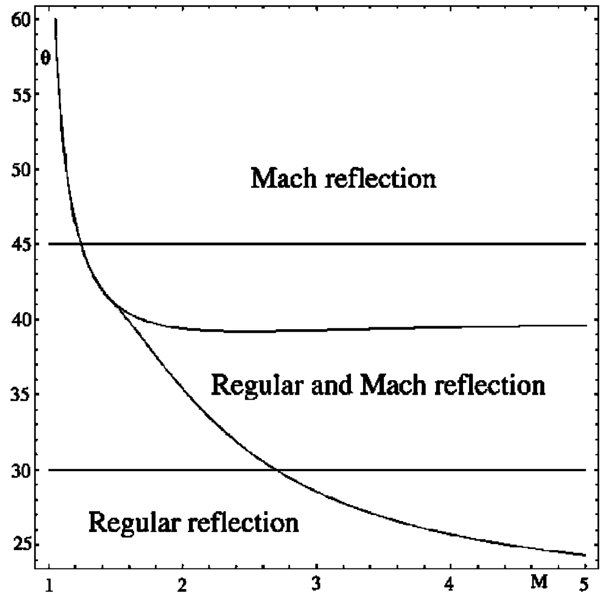

FIG. 2. Transition criteria for regular reflection to Mach reflection for $\gamma = 1.4$. The horizontal line at $\theta = 30$ represents the triangular self-similar shock waves. The line at $\theta = 45$ represents the square shocks.

Fig. 2 (representing the detachment criterion) can have only a Mach reflection. When (M, θ) lies below the lower curve, the regular reflection is stable according to the mechanical equilibrium criterion. Between the two curves, both regular reflection or Mach reflection are possible [13]. For $\gamma = 1.4$ and incidence angle $\pi/6$ (30°), the reflection is stable under the mechanical equilibrium criterion if $M < 2.695\,06$, and is stable under the detachment criterion for all $M > 1$. Thus, for $\gamma = 1.4$ our triangular convergent solution is stable for $M < 2.695\,06$. For larger Mach numbers, the solution may still be stable according to the detachment criterion, and whether this is the case or not may depend on the details of the initial conditions.

The construction of the self-similar equilateral triangular shock can be extended to nonequilateral triangles by considering three convergent shocks with the same Mach number. In this case, the reflection angles may be different at each corner. Indeed, self-similar solutions can also be constructed for other polygonal shapes besides the triangle, but, in general, these solutions will tend to be unstable. The reason is that the incidence angle θ will be larger, and a Mach reflection will appear at the corners. For example, for a square shock and $\gamma = 1.4$, the incidence angle is $\pi/4$, and regular reflection will be possible only for $M < 1.24$ according to the detachment criterion. For larger Mach numbers, Mach reflection always occurs. For regular polygons with five or more sides, there is an additional reason that excludes the self-similar regime: The reflected shocks cross each other making our solution invalid. This can be easily seen in the acoustic limit $M = 1$, in which the reflected angles are equal to the incidence angles. When the self-similar solutions are not possible, the focusing will presumably proceed according to the alternating polygonal solution of Schwendeman and Whitham [9].

We now show that suitable perturbations of a circular shock will lead to the self-similar focusing given by the triangular shock. We implemented a Godunov-based method to integrate the compressible gas-dynamics equations and study the initial value problem for a convergent flow. We used a slope-limited monotone upstream-centered scheme for conservation laws (MUSCL)-Hancock scheme with dimensional splitting and with an exact Riemann solver to compute the fluxes [14]. We validated the numerical scheme with exact solutions of the 1D Riemann problem in which the waves form an angle of 30° with respect to the numerical grid, with the Sedov solution for a strong blast [15], and with the exact solution for oblique regular reflection of shock waves.

We studied convergent shocks in a gas with $\gamma = 1.4$. We used the Riemann-like initial condition $p(r) = 1$, $\mathbf{v}(r) = 0$ and $\rho(r) = 1$ on $r > R_0(1 + \epsilon \cos 3\theta)$, and $p(r) = 0.01$, $\mathbf{v}(r) = 0$ and $\rho(r) = 0.1$ for $r < R_0(1 + \epsilon \cos 3\theta)$. We set $R_0 = 0.1$, $\epsilon = 0.1$, and compute the solution on the domain $[-1, 1] \times [-1, 1]$. The domain is discretized with a grid of 600×600 points. On the boundary of the domain we use transmission [14] boundary conditions (normal derivatives of p, ρ, and $\mathbf{v}$ are zero at the boundary).

In order to accurately compute the flow near the incident shock, we apply a dynamical gridding procedure similar to the one used in [16]. Every time that the area C enclosed by the incident shock is reduced by a factor 2, we interpolate the solution in a second grid that is smaller by a

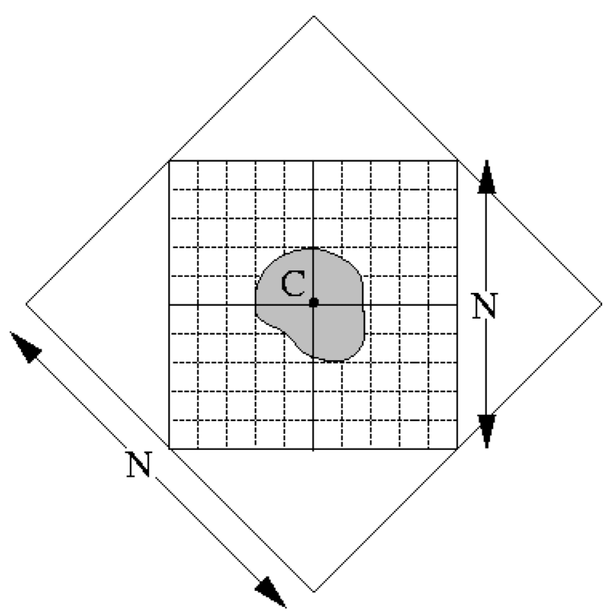

FIG. 3. Every time the area enclosed by the incident shock reduces a factor 2, the central part of the domain of integration is regridded by maintaining the same number N of discretization points.

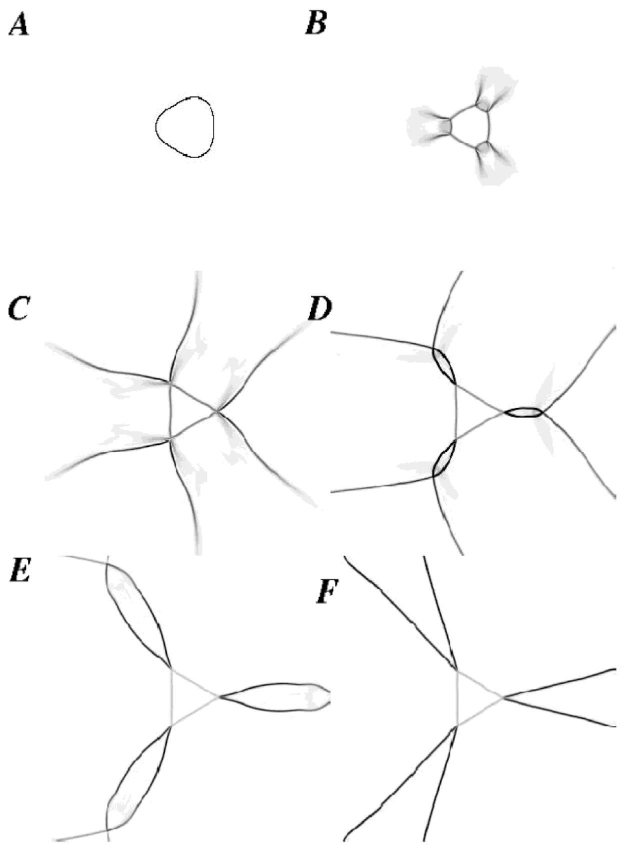

FIG. 4. Simulated schlieren of the initial condition, and after 5, 9, 11, 13, and 17 regriddings (in lexicographical order).

factor $\sqrt{2}$ and that has the same number N of gridpoints, as depicted in Fig. 3. The points outside the new grid are discarded.

In Fig. 4 we show the resulting evolution of the shocks. The grey scale of these pictures is proportional to the absolute value of the gradient of the density. On every snapshot, the coordinates were normalized with the square root of the area bounded by the incident shock. Figure 4A is the initial condition. In stage 4B the area reduced by a factor 2^5, and a Mach stem appears on the incident shock while new shocks are generated behind it. These new shocks propagate laterally and cross in 4D. Afterwards, the self-similar solution is gradually approached. In order to observe the self-similar regime in this experiment, the area enclosed by the incident shock has to be at least 2^{17} times smaller than the initial area. The asymptotic Mach number is $M = 6.5$. We verified that the finite size of the computational domain does not affect the results by making two additional computations with $R_0 = 0.2$ and 300×300 gridpoints, and $R_0 = 0.05$ and 1200×1200 gridpoints on the domain $[-1, 1] \times [-1, 1]$, and verifying that we obtain the same triangular asymptotics. We also repeated the computations for $\gamma = 5/3, 3/2$, and $4/3$ and found the same self-similar asymptotic behavior. For $\gamma = 2$ we obtain the triangular solution for $\epsilon = 0.2$, and the alternating solution for $\epsilon = 0.1$.

Although we have shown that some initial perturbations of a circular shock of the form $r = R_0(1 + \epsilon \cos 3\theta)$ will lead to self-similar triangular focusing, it is natural to ask for a more comprehensive characterization of the initial perturbations which lead to triangular asymptotics. For example, if $r = R_0(1 + 0.1 \cos 3\theta + 0.02 \cos 5\theta)$, we still get self-similar triangular asymptotics, but the triangles are not equilateral. A systematic investigation of this question will require a significant improvement of our numerical scheme. At present we are limited to about 20 regridding operations, but it appears that many more will be required if we hope to distinguish between self-similar and alternating focusing from multimode perturbations.

Other open problems are the detailed stability analysis of the self-similar solution and the study of the possibility of generalizations to three dimensions.

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