Comment on "Quantum Games and Quantum Strategies"

In a recent Letter, Eisert *et al.* [1] introduce an elegant scheme for quantizing classical games, and proceed to perform an extensive analysis of its application to the famous two-player game, Prisoner's Dilemma. In the traditional form of this game, rational analysis leads the players to "defect" against one another in a mutually destructive fashion [2]. A central result of the Letter of Eisert *et al.* is the observation that "by allowing quantum strategies the players escape the dilemma"—a new equilibrium replaces the mutual defection. Here we argue that Eisert *et al.* achieve this result only by applying an artificial constraint on the set of strategies available to the players: they are permitted a certain strategy, $\hat{\sigma}_z$, and yet they are forbidden the logical counterstrategy, $\hat{\sigma}_x$. We explicitly show that, when the players are permitted free choice of any unitary strategy, the behavior of the game is indeed wholly different.

The players of the quantum game are allowed a set of "strategies" S that correspond to "some subset of the group of unitary 2×2 matrices." The only fundamental restriction on the choice of S is that it should include the identity, $\hat{I}$, and the Pauli matrix, $\hat{\sigma}_y$, which correspond to "cooperate" and "defect" in the traditional game. Eisert *et al.* specify their choice of S in the following sentence: "it proves to be sufficient to restrict the strategic space to the two-parameter set of unitary 2×2 matrices, $[\hat{U}(\theta,\phi) = \cos(\theta/2)\exp(i\phi\hat{\sigma}_z) + i\sin(\theta/2)\hat{\sigma}_y]$, with $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq \pi/2$." There appear to be two possible meanings of the word "sufficient": (a) using S is sufficient to discover properties of the general quantum game, or (b) using S is sufficient to create some feature not seen classically.

While Eisert *et al.* appear to have intended meaning (b), their original paper did not make this clear; all the readers to whom we have spoken have instead understood meaning (a), which, as we will show, is incorrect. Furthermore, given that (b) was the intended meaning, one would expect some discussion of the choice of S—otherwise one cannot judge the significance of the result. To us it seems unlikely that restricting the players to this set can reflect any reasonable physical constraint (limited experimental resources, say), because the set is not closed under composition. The (forbidden) ideal counterstrategy to $\hat{\sigma}_y$ is $\hat{\sigma}_x$, which is equal to two consecutive allowed manipulations: $\hat{U}(0,\pi/2)\hat{U}(\pi,0)$. The authors' choice of S therefore appears difficult to justify in its own right—rather it must be seen as a construct designed to demonstrate that new features can occur for *some* choice of S. But the existence of such features is not surprising—in fact, it can be established very easily by using the trivial discrete set $S = \{\hat{I}, \hat{\sigma}_y, \hat{\sigma}_z\}$. We note that $\hat{\sigma}_y$ is the ideal counterstrategy if one's opponent plays $\hat{I}$, and similarly $\hat{\sigma}_z$ is the ideal counter to $\hat{\sigma}_y$. However, the ideal counter to $\hat{\sigma}_z$, namely $\hat{\sigma}_x$, is forbidden. Thus, as one might expect, the players gravitate toward using $\hat{\sigma}_z$, and the game has a Nash equilibrium in which both players do so. It is also unsurprising that when we remove the unnatural restriction and offer the full discrete set $\{\hat{I}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$ that the game changes completely—we find that every strategy has a perfect counterstrategy ($\hat{I}$ counters $\hat{\sigma}_x$); thus there is no equilibrium. The strategy set employed by Eisert *et al.* is simply the continuous analog of our restricted discrete set, and, as we show below, the exactly analogous effect occurs when we lift their restriction.

We will write the operations applied by the players in the form $\hat{X} \otimes \hat{Y}$, where $\hat{X}$ is applied to the qubit controlled by A and $\hat{Y}$ to that controlled by B. Suppose that player A applies $\hat{X}$ to her qubit, prepared as the first in the entangled state $\hat{J}|CC\rangle$, where $\hat{J} = \exp\{i\pi\hat{D} \otimes \hat{D}/4\}$ and $\hat{D} = i\hat{\sigma}_y$ is the defect matrix of [1]. The most general $\hat{X} \in$ SU(2) is of the form $\hat{X} = (x_{ij})$, where $x_{11} = x_{22}^*$, $x_{12} = -x_{21}^*$, and $\det\hat{X} = 1$. Therefore, A produces the state $(\hat{X} \otimes \hat{I})\hat{J}|CC\rangle = (\hat{I} \otimes \hat{Y})J|CC\rangle$ for $\hat{Y} = (y_{ij}) \in$ SU(2), where $y_{11} = x_{11}$ and $y_{12} = ix_{12}$. In other words, any unitary transformation which A applies locally to her qubit is actually equivalent to a unitary transformation applied locally by B. Consequently, if B were to choose $\hat{D}\hat{Y}^\dagger$, we would have a final state $\hat{J}^\dagger(\hat{X} \otimes \hat{D}\hat{Y}^\dagger)\hat{J}|CC\rangle = \hat{J}^\dagger(\hat{I} \otimes \hat{D}\hat{Y}^\dagger\hat{Y})\hat{J}|CC\rangle = |CD\rangle$, the optimal outcome for B. Thus, for any given strategy of A, there is an ideal counterstrategy for B, and vice versa. Not only does the pair of strategies found by Eisert *et al.* fail to form a Nash equilibrium in the space of unitary strategies; in this space there are no Nash equilibria.

Equilibria *are* seen if we go beyond the unitary space, e.g., to the space of all physically possible moves. Eisert *et al.* mention this possibility in a footnote, but the body of their Letter is entirely concerned with the unitary game. It seems to us that equilibria can be obtained only in the maximally entangled two-player [3] unitary game when the strategies are unnaturally constrained. This is not the impression one gains from reading the Letter.

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[1] J. Eisert, M. Wilkens, and M. Lesenstein, Phys. Rev. Lett. **83**, 3077 (1999).
[2] R. B. Myerson, *Game Theory: An Analysis of Conflict* (MIT Press, Cambridge, MA, 1991).
[3] Pure unitary equilibria *do* occur in multiplayer games. See S. C. Benjamin and P. M. Hayden, e-print quant-ph/0007038 [Phys. Rev. A (to be published)].

