

Class of Exactly Solvable Pairing Models

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We present three classes of exactly solvable models for fermion and boson systems, based on the pairing interaction. These models are solvable in any dimension. As an example we show the first results for fermions interacting with repulsive pairing forces in a two-dimensional square lattice. In spite of the repulsive pairing force the exact results show attractive pair correlations.

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Exactly solvable models have played an important role in understanding the physics of the quantum many body problem, especially in cases where the system is strongly correlated. Such situations arise, e.g., in one-dimensional (1D) systems of interest for condensed matter physics and also in strongly correlated finite fermion systems as atomic nuclei. In both branches of physics the study of exactly solvable models has been pursued with enormous success.

In 1D quantum physics, the exactly solvable models can be classified into three families. The first family began with Bethe's exact solution of the Heisenberg model. Since then a wide variety of 1D models has been solved using the Bethe ansatz (for a recent review see [1]). A second family of models is the so-called Tomonaga-Luttinger models [1] which are solved by bosonization techniques and which revealed non-Fermi liquid properties in 1D. These systems are now called Luttinger liquids. The third family, proposed by Calogero and Sutherland, is models with long-range interactions. They have been applied to several problems [1] such as spin systems, the quantum Hall effect, random matrix theory, etc.

The impact of the exactly solvable models in condensed matter physics and in nuclear physics is so enormous that one can hardly believe that the exact solution of the pairing model (PM), of great interest to both fields, passed almost unnoticed until very recently [2]. The PM was solved exactly by Richardson in a series of papers in the 1960s [3–5].

Independent of Richardson's exact solution, it was recently demonstrated [6] that the PM is an integrable model. The pairing model may turn out particularly interesting, since recent work [7] has shown that the pure repulsive pairing Hamiltonian in a 2D lattice can be solved exactly in the thermodynamic limit revealing strong superconducting fluctuations. The importance of this finding stems, of course, from the fact that high- T_c superconductors apparently acquire their superconducting properties through the repulsive Coulomb interaction.

We will derive in this Letter three families of exactly solvable models based on the pairing interaction for fermion systems as well as for boson systems. The most

important feature of the new set of models we will present is that they are exactly solvable in any dimension. In [8] we have advanced a numerical solution for a three-dimensional confined boson system; here we will give preliminary results for a fermion system in a 2D lattice.

Since the proof of integrability and the derivation of the exact solutions are completely analogous for fermions and bosons, we will develop both systems in parallel. In what follows, whenever there are different signs, the upper one will correspond to bosons while the lower one to fermions, and we will refer indistinctly to bosons and fermions as particles.

Let us begin our derivation by defining the three operators

$$\hat{n}_l = \sum_m a_{lm}^\dagger a_{lm}, \quad A_l^\dagger = \sum_m a_{lm}^\dagger a_{l\bar{m}}^\dagger = (A_l)^\dagger, \quad (1)$$

which close the commutator algebra

$$[\hat{n}_l, A_l^\dagger] = 2\delta_{ll'} A_l^\dagger, \quad [A_l, A_l^\dagger] = 2\delta_{ll'} (\Omega_l \pm 2\hat{n}_l). \quad (2)$$

In Eq. (1) the pair operator A_l^\dagger creates a pair of particles in time reversal states with a^\dagger (a) the particle creation (annihilation) operator and Ω_l being the degeneracy of level l .

The number operator \hat{n}_l and the pair operators A_l, A_l^\dagger in each level l close the commutator algebra of the groups SU(2) for fermions or SU(1, 1) for boson systems.

The three generators of these algebras can be written in terms of the previously defined pair and number operators as $K_l^0 = \frac{1}{2}\hat{n}_l \pm \frac{1}{4}\Omega_l$, and $K_l^\pm = \frac{1}{2}A_l^\dagger = (K_l^\mp)^\dagger$.

These generators obey the more familiar commutation relations of the SU(1, 1) and SU(2) group algebras

$$[K_l^0, K_l^\pm] = \delta_{ll'} K_l^\pm, \quad [K_l^+, K_l^-] = \mp 2\delta_{ll'} K_l^0. \quad (3)$$

The difference between the algebra of SU(1, 1) and SU(2) appears in the sign of the second commutator.

The Hilbert space of N particles moving in L single particle levels can be classified according to the product of groups SU(2)₁ × SU(2)₂ × ... × SU(2)_L for fermions or SU(1, 1)₁ × SU(1, 1)₂ × ... × SU(1, 1)_L for bosons.

A complete set of states in this Hilbert, which are space spanned by the pair operators (1), is given by

$$|n_1, n_2, \dots, n_L, \nu\rangle = \frac{1}{\sqrt{\mathcal{N}}} A_1^{\dagger n_1} A_2^{\dagger n_2} \dots A_L^{\dagger n_L} |\nu\rangle, \quad (4)$$

where \mathcal{N} is a normalization constant. The possible number of pairs in each level is $0 \leq n_l \leq \Omega_l/2$ for fermion systems or $0 \leq n_l \leq N/2$ for boson systems. A state $|\nu\rangle \equiv |\nu_1 \nu_2 \dots \nu_L\rangle$ of unpaired particles is defined as

$$A_l |\nu\rangle = 0, \quad \hat{n}_l |\nu\rangle = \nu_l |\nu\rangle, \quad K_l^0 |\nu\rangle = d_l |\nu\rangle, \quad (5)$$

where $d_l = (\frac{\nu_l}{2} \pm \frac{\Omega_l}{4})$ and $N = 2M + \nu$, M being the number of pairs and ν the total number of unpaired particles. We will borrow from nuclear physics [9] the name seniority for the number of unpaired particles in each level ν_l .

In the product spaces mentioned above a model is integrable if there are L independent global operators commuting with one another. These operators are the quantum invariants and their eigenvalues are the constants of motion of the system. In looking for these operators, let us define the most general combination of one and two body operators in terms of the K generators with the condition of being Hermitian and number conserving:

$$R_l = K_l^0 + \left\{ 2g \sum_{l'(\neq l)} \frac{X_{ll'}}{2} (K_l^+ K_{l'}^- + K_l^- K_{l'}^+) \mp Y_{ll'} K_l^0 K_{l'}^0 \right\}. \quad (6)$$

Up to now the matrices X and Y are completely free, but we will fix them imposing the condition that the R operators should commute among one another to define an integrable model. The condition $[R_l, R_{l'}] = 0$ will be fulfilled if they are antisymmetric and satisfy the following equation:

$$Y_{ij} X_{jk} + Y_{ki} X_{jk} + X_{ki} X_{ij} = 0. \quad (7)$$

An analogous condition has been encountered by Gaudin [10] in a spin model known at present as the Gaudin magnet. His model is based on R operators similar to (6) but without the one body term. He found three different solutions for the condition (7) which can be grouped together in compact form as $X_{ll'} = \frac{\gamma}{\sin[\gamma(\eta_l - \eta_{l'})]}$, $Y_{ll'} = \gamma \cot[\gamma(\eta_l - \eta_{l'})]$, where the different models are distinguished by the value of γ .

I. The rational model: $\gamma \rightarrow 0$

$$X_{ll'} = Y_{ll'} = \frac{1}{\eta_l - \eta_{l'}}. \quad (8)$$

II. The trigonometric model: $\gamma = 1$

$$X_{ll'} = \frac{1}{\sin(\eta_l - \eta_{l'})}, \quad Y_{ll'} = \cot(\eta_l - \eta_{l'}). \quad (9)$$

III. The hyperbolic model: $\gamma = i$

$$X_{ll'} = \frac{1}{\sinh(\eta_l - \eta_{l'})}, \quad Y_{ll'} = \coth(\eta_l - \eta_{l'}). \quad (10)$$

η_l is an arbitrary set of nonequal real numbers. Each solution gives rise to an integrable model and any combination of the R operators produces an integrable Hamiltonian. Since the three models have quite different properties, we prefer to continue our derivation in an independent way. It is worthwhile to mention here that if we relax the condition of the number conservation on the set of operators (6) there is a general solution in terms of elliptic functions [10].

The rational model has been recently proposed in Ref. [6] to demonstrate the integrability of the PM Hamiltonian. Indeed the PM Hamiltonian can be obtained from the rational model by means of linear combination of R operators $H_P = 2 \sum_l \eta_l R_l^{\dagger}$ plus an appropriate constant to give

$$H_P = \sum_l \varepsilon_l \hat{n}_l + \frac{g}{2} \sum_{ll'} A_l^{\dagger} A_{l'}, \quad (11)$$

where the free parameters η_l have been replaced by the single particle energies ε_l .

Then, the PM Hamiltonian is diagonal in the basis of common eigenstates of the R operators. But, also, any arbitrary linear combination of the R operators defines a valid integrable Hamiltonian. In the latter case, the total number of free parameters in defining the Hamiltonian is $2L + 1$ for fermion as well as for boson systems.

We would like to emphasize that the demonstration of integrability given above for the three models does not imply that they are exactly solvable, i.e., that the complete set of common eigenstates of the R operators can be obtained. As mentioned before, Richardson only worked out the eigenstates for the PM Hamiltonian and, as a matter of fact, the authors of Ref. [6] were not aware of the existence of that exact solution. Very recently [11] the eigenvalues of the R operators of the rational model for fermion systems were obtained, within a conformal field theory formalism, in the fully paired subspace ($\nu_l = 0$) and with constant degeneracy $\Omega_l = 2$.

To begin our derivation we will propose an ansatz for the eigenstates of the R operators in the Hilbert space of states (4) which is a generalization of the ansatz used by Richardson to find the eigenstates of the PM. The exact eigenstates for the three models can be written as product pair wave function acting on the space of unpaired particles $|\nu\rangle$

$$|\Psi\rangle = \prod_{\alpha=1}^M B_{\alpha}^{\dagger} |\nu\rangle, \quad B_{\alpha}^{\dagger} = \sum_l u_l(e_{\alpha}) K_l^{\dagger}. \quad (12)$$

The function u depending on the pair energies e has the form of the eigenstate of the one pair problem in each of the three models, and the set of parameters e is left as free parameters to be fixed in order to fulfill the L eigenvalue equations

$$R_l |\Psi\rangle = r_l |\Psi\rangle. \quad (13)$$

The collective amplitudes in the pair operators B for each model are

$$u_{\alpha i}^I = u_i(e_\alpha) = \frac{1}{2\eta_i - e_\alpha}, \quad (14)$$

$$u_{\alpha i}^{II-III} = u_i(e_\alpha) = \frac{1}{\text{sn}(e_\alpha - \eta_i)}. \quad (15)$$

We will treat on equal footing the trigonometric and the hyperbolic models to solve the eigenvalue equation (13). To embody both derivations in the same formalism we will use the symbols sn for \sin or \sinh , cs for \cos or \cosh , and ct for \cot or \coth , not to be confused with elliptic functions. Note that we have already used sn in (15).

Here we summarize the final results for the three models leaving the details of the derivation for a future publication.

Model I:

$$1 \pm 4g \sum_j \frac{d_j}{2\eta_j - e_\alpha} \mp 4g \sum_{\beta(\neq\alpha)} \frac{1}{e_\alpha - e_\beta} = 0, \quad (16)$$

$$r_i = d_i \left[1 \mp 2g \sum_{j(\neq i)} \frac{d_j}{\eta_i - \eta_j} \mp 4g \sum_\alpha \frac{1}{2\eta_i - e_\alpha} \right]. \quad (17)$$

Models II and III:

$$1 \mp 2g \sum_j d_j \text{ct}(e_\alpha - \eta_j) \pm 2g \sum_{\beta(\neq\alpha)} \text{ct}(e_\beta - e_\alpha) = 0, \quad (18)$$

$$r_i = d_i \left\{ 1 \mp 2g \left[\sum_{j(\neq i)} d_j \text{ct}(\eta_i - \eta_j) - \sum_\alpha \text{ct}(e_\alpha - \eta_i) \right] \right\}. \quad (19)$$

In order to obtain the pair energies e_α , given a set of parameters η and a pairing strength g , one has to solve the coupled set of M nonlinear equations (16) for the rational model or (18) for the trigonometric or hyperbolic models, respectively. In the limit $g \rightarrow 0$ one immediately realizes that the equations (16), (18) can be fulfilled only for $e_\alpha \rightarrow 2\eta_i$. Then the amplitudes $u_{\alpha i}$ in (14), (15) become diagonal and we see that the states (12) are equivalent in this limit to the complete set of uncorrelated states (4). From this $g \rightarrow 0$ limit one can construct the ground state (with pairs filling the lowest possible states), and the configurations of the successive excited states. For example, the first excited state is obtained by promoting the highest energy pair to the next upper 2η value, or by breaking a pair (removing a pair energy) into two unpaired particles (increasing the seniority ν by 2). One then follows the trajectory of the pair energies e_α for each of the configurations as a function of g solving the equations (16), (18).

For boson systems the pair energies stay always real, but for fermion systems the pair energies can be either real or complex conjugate pairs. In the latter case there

might arise singularities in the solution of the equations (16), (18) for some critical value of the pairing strength g_c for which two or more pair energies acquire the same value. It can be shown [12] that each one of these critical g values is related to a single particle level i and that at the critical point $1 - 2d_i$ pair energies must be equal to $2\eta_i$. These singularities cancel out in the calculations of energies which do not show any discontinuity in the vicinity of the critical points.

The eigenvalues of the R operators, given by (17) or (19), respectively, are always real since the pair energies are real or complex conjugate pairs. Each solution of the nonlinear set of equations produces an eigenstate common to all R_i operators, and consequently to any Hamiltonian written as a linear combination of them. The corresponding eigenvalue is the linear combination of the r_i eigenvalues. As mentioned before, the most important feature of the three families of models is that they are exactly solvable in any dimension. The dimensionality enters through the degeneracies $d_i = \nu_i/2 - \Omega_i/4$. The information about states with nonzero seniority (broken pairs) also enters through the same term. Symmetry breaking terms, such as anisotropic hoppings or disorder, might lift up the degeneracies but still leave the Hamiltonian as exactly solvable. In such cases the dimensionality would show up in the density of states.

Next we will present the first results for the rational model of fermions in a 2D square lattice of $P \times P$ sites with periodic boundary conditions and a repulsive pairing interaction. Assuming a restricted hopping term between nearest neighbors, the single fermion levels are $\varepsilon_k = -2(\cos k_x + \cos k_y)$, with $k_\sigma = 2\pi n_\sigma/P$ and $-P/2 \leq n_\sigma < P/2$. Here we will consider a 6×6 lattice at half filling ($M = 18$) with a PM Hamiltonian for which $\eta_i = \varepsilon_i$ in (16), but the properties we will discuss are quite general and independent of the latter choice.

The single fermion energies ε_k and the corresponding degeneracies Ω_k are displayed in Table I. In the limit $g = 0$ the ground state is obtained by distributing the $M = 18$ pairs in the lowest possible states.

In Fig. 1 we show the real and imaginary parts of the pair energies. Only the real part and the positive imaginary part are shown for each complex conjugate pair. The first level accommodates one pair which is forced to be real. The next three levels accommodate four pairs each, forming two pairs of complex conjugates in each level. In the last level we can put five pairs of particles forming two pairs of complex conjugates and one real pair energy. For the critical value $g_1 = 0.1708$ the first five pairs become equal to $2\varepsilon_2 = -6$ as discussed before. As seen in Fig. 1,

TABLE I. Single particle energies and degeneracies for the 6×6 lattice.

ε_j	-4	-3	-2	-1	0	1	2	3	4
Ω_j	2	8	8	8	20	8	8	8	2

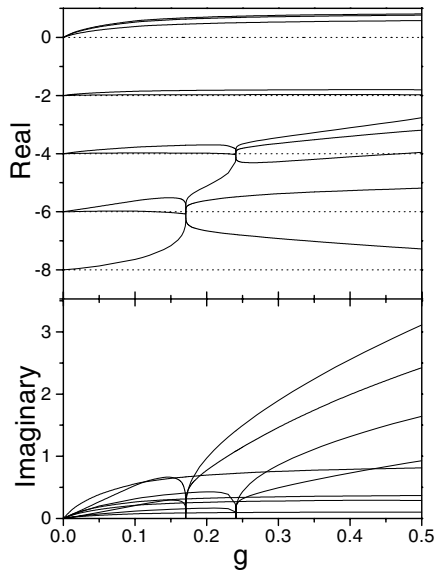


FIG. 1. Real and positive imaginary parts of the pair energies e_α for a 6×6 lattice at half filling as a function of g .

the real part of the first two complex conjugated pairs together with the first real pair energies crosses at -6 while the imaginary parts go to zero. A similar situation arises for the second critical point at $g_2 = 0.2407$ in which five pairs become equal to $2\varepsilon_3 = -4$.

In spite of the quite involved behavior of the pair energies, the total energy displays a smooth behavior as a function of g . In Table II we give some values of the correlation energy density ($[E(g) - E_{HF}(g)]/P^2$) and the interaction energy density ($[E(g) - E(0)]/P^2$) as a function of g . The interaction energy increases with increasing values of g , showing a tendency to saturation that is consistent with its vanishing in the thermodynamic limit [13]. The correlation energy density decreases almost linearly with g . These attractive pairing correlations based on a repulsive interaction give further numerical support to the work of Shastry [7] who found quasi-long-ranged order in the thermodynamic limit of the model at half filling.

In summary, we have presented three families of new exactly solvable models based on the pairing interaction for fermion and boson systems. These models have the important feature of being solvable in any dimension. We have presented preliminary results for the properties of

TABLE II. Correlation energy densities and interaction energy densities for various values of g for the 6×6 lattice at half filling.

g	0.0	0.1	0.2	0.3	0.4	0.5
E_{corr}	0.0	-0.74	-1.55	-2.37	-3.19	-4.03
E_{int}	0.0	0.12	0.18	0.22	0.25	0.28

the exact solution of the rational model in a 2D square lattice with repulsive pairing interaction. This model may be useful to study features of high- T_c superconductivity because, in spite of the purely repulsive bare force, the exact solution shows attractive pair correlations.

After completing this work we learned of a recent preprint [14] in which the hyperbolic model is presented for electrons in ultrasmall superconducting grains, however without applications. This model is equivalent to our models II and III for fermions in 1D with seniority zero.

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