Random versus Realistic Interactions for Low-Lying Nuclear Spectra

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We compare the shell-model results for realistic interactions with those obtained for various ensembles of random matrix elements. We show that, although the quantum numbers of the ground states in the even-even nuclei have a high probability (∼60%) to be $J^\pi T = 0^+0$, the overlap of those states with the realistic wave functions is very small in average. The transition probabilities $B(E2)$ predicted with random interactions are also too small. The presence of the regular pairing is shown to be a significant element of realistic physics not reproduced by random interactions.

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The interplay of regular and chaotic elements in quantum many-body dynamics was extensively studied in the framework of random matrix theory [1,2] and in realistic models of atoms, nuclei, condensed matter, and quantum fields (see, for example, [3–6]). The embedded Hamiltonian ensembles and, in particular, the two-body random Hamiltonian ensembles [1,7–9] are especially relevant for these studies. The existence in finite systems of exact conservation laws, such as angular momentum, parity, or isospin, raises new questions, for instance, how quantum chaos is influenced by these indestructible symmetries, and what the correlations are (if any) between the blocks of states with different exact quantum numbers governed by the same Hamiltonian. Such questions are important for understanding the nature of observed regularities in actual many-body spectra.

The nuclear shell model with the effective two-body forces in a restricted Hilbert space is the best available theoretical tool for calculating the properties of the low-lying states. Recently [9–12], the low-lying spectra were studied with the shell-model techniques but using, instead of effective interactions, randomly generated (but rotationally invariant) two-body matrix elements. Some of the results resemble the pattern of actual nuclear spectra. One particularly interesting observation was that of predominance of spin $J = 0$ in the ground state in spite of the low statistical weight of states with $J = 0$ in Hilbert space. This result is robust and insensitive to precise statistical properties of the random interaction in the fermion shell model [13] or in the interacting boson approximation [14,15]. A simple mechanism of random coupling of individual particle spins was suggested in Ref. [12] to explain the preponderance of $J = 0$ (and, in some cases, $J = J_{\max}$ [12,14]) in the ground state. In average, the yrast line in a randomly interacting fermionic system acquires a random sign of the effective moment of inertia which leads to the large probabilities of the edge values of the total spin.

In this Letter, we study the wave functions resulting from the shell-model calculations with realistic and random interactions. We show that the overlap of the 0^+ ground-state (gs) wave functions generated by random interactions with those obtained for realistic interactions is very small. Also, the associated transition probabilities $B(E2)$ to the 2_1^+ state are very small. The implication is that the order which is present in actual nuclear states is almost entirely due to the coherent (nonrandom) aspects of the nuclear Hamiltonian. We need to stress that here we look for the signatures of the coherent phenomena not in the spin ordering which might be a consequence of geometrical constraints, but in the collectivity of the wave functions. As shown by many authors, the highly excited states in real systems are very complex and in many aspects similar to the eigenfunctions of a random Hamiltonian. However, the degree of complexity is systematically evolving along the spectrum [4] so that the low-lying states are much more regular.

The experience with the shell-model calculations demonstrated that there exists a rather well determined set of single-particle energies and two-body interaction matrix elements which, being processed through the machinery of the large scale shell-model diagonalization, lead to a good description of the low-lying spectra in agreement with data [16]. As a generic system we take that of eight particles in the *sd* shell, the case corresponding to the well studied ^{24}Mg nucleus. The geometry of the system is much richer than that of the schematic single- j model studied in Ref. [12]; it includes also isospin variables. This allows us to draw some conclusions concerning actual nuclear structure. In the *sd* model space, there are 63 independent matrix elements under constraints of rotational and isospin invariance. We use two interactions, one from Ref. [16] and SDPOTA from Ref. [17], denoted below as (W) and (P), respectively. Being based on different approaches [a fit of individual matrix elements (W) and a fit of a potential (P)], these sets agree in predicting the ground state with quantum numbers $J^\pi T = 0^+0$. The ground-state wave functions for the two interaction sets overlap by 98%. Our conclusions are the same for both realistic interactions, and the figures below will show the results for (W).

Earlier it was suggested [10] that the wave functions generated by the random interactions carry significant pairing correlations. Having this conjecture in mind, four models of random interactions were considered: (a) degenerate single-particle energies $0d_{5/2}, 1s_{1/2}, 0d_{3/2}$ (set to zero); all 63 two-body matrix elements generated as random variables uniformly distributed in the interval $[-1, 1]$ (in this case, the energy scale is arbitrary); (b) single-particle energies taken from the realistic interaction, W or P, while 63 two-body matrix elements were uniformly generated in the interval $(a - s, a + s)$, where a is the average of the magnitude of the matrix elements in the interaction (W or P), and $s = \sqrt{3}\,\sigma$; σ is the variance of the matrix elements in the corresponding realistic interaction; the values of a and s are -0.818 and 3.12 MeV, respectively; (c) the isospin-invariant pairing, $JT = 01$, two-body matrix elements were kept from the realistic interactions, whereas the remaining matrix elements were generated as in (b) with the values of a and s equal to -0.616 and 3.03 MeV, respectively; (d) only six pairing, $JT = 01$, two-body matrix elements were randomly generated in the interval $[-1, 1]$, whereas all other matrix elements and all single-particle energies were set to 0; with respect to the energy value, this case can be compared to (a).

The results for 1000 realizations for each model are combined in Table I. The ground-state energy (relative to ^{16}O and corrected for Coulomb energy as in [16]) with both realistic interactions is -87.1 MeV. The average ground-state energy in cases (b) and (c) is of the same order as in realistic calculations being mainly determined by the single-particle energies. The gain of 2.9 MeV in version (c) compared to (b) is related to the realistic pairing. The loss of 5 MeV in (b) compared to the full realistic interaction is due to pairing plus multipole-multipole correlations present in the realistic case. The average positions of the ground state in purely random versions (a) and (d) are determined by the widths of the Gaussian many-body level densities known for many-body systems with a random two-body interaction [1,4].

For all random ensembles, the predominance of the ground states with $J = 0$ is seen clearly. In the fully random case (a), the result is in agreement with what was observed in the pioneering paper [9], confirmed in later studies, and attributed mainly to the random geometrical coupling [12]. The presence of regular pairing, case (c), increases the percentage of the ground states with $J = 0$. The strongest effect is observed for the case (d) when the off-diagonal pair transfer matrix elements make quantum numbers $J = T = 0$ preferable for an even number of pairs, as in the case under study, while the competing influence of incoherent interactions is absent.

The average overlaps $\langle W | R\rangle^2 = |\sum_k C_k^{gs}(W) C_k^{gs}(R)|^2$ of the ground-state wave function for the W and P interactions with the 0^+ ground states of the four different models of the random interaction are presented, along with their variances, in Table I. The average overlap is small in all cases; in particular, in the case (a), 2.0%. This is in agreement with the conclusion drawn in Ref. [12] for a single-j level model that the ground-state wave function carries very little effect of pairing correlations. However, this overlap is still greater than one would expect in the case of extreme chaoticity when the components C_k^{α} of a generic wave function $|\alpha\rangle$ are uniformly distributed over the unit sphere in space of the corresponding dimension N and $\overline{|C|^2} = 1/N$ which gives rise to the so-called N scaling [18,19]. In our case, the dimension for the $J = 0, T = 0$ states is $N = 325$ which would give the average chaotic overlap factor 0.3%.

The average overlap is even greater in other models. The maximum of 11% is reached in model (c) because of the combined action of two effects. First, the presence of realistic pairing lowers the energy of a state with paired particles. On the other hand, basis states with large seniority (the number of unpaired particles) are now effectively removed from contributing considerably to the ground-state wave functions. This makes the effective dimension N smaller than the nominal one. This phenomenon was clearly seen for a simple $N = 3$ single-j case in Ref. [12]. The stabilizing presence of the mean-field orbitals, model (b), also increases the overlap with the realistic ground-state wave function. The models (a) and (d) have overlaps which are strongly peaked at small values. The model (c) leads to a more smooth and uniform

TABLE I. Results for models (a), (b), (c), and (d) as described in the text.

		(a)	(b)	(c)	(d)
	% of $JT = 00$ gs	59.1	49.3	67.8	92.2
	$\langle E_{00}\rangle$ (MeV)	-13.8	-82.1	-85.0	-3.6
(W)	Average $\langle W \| R\rangle^2$	0.020	0.053	0.106	0.052
	Variance $\langle W \| R\rangle^2$	0.056	0.094	0.137	0.096
(P)	Average $\langle P \| R\rangle^2$	0.019	0.054	0.113	0.061
	Variance $\langle P \| R\rangle^2$	0.051	0.088	0.137	0.113
	$B(E2)_{av}$	7.0	9.9	14.3	6.2
	$\sigma_{B(E2)}$	10.5	11.9	15.5	5.0
	$B(E2)_{max}$	59.2	59.4	68.0	22.1

overlap distribution. This again shows that the realistic mean-field orbitals (given by their single-particle energies) have a strong influence on the overlap. The overlap is more enhanced by realistic pairing (c), but it is still far from unity since other coherent correlations are missing.

The complexity of the eigenfunctions in a given basis $|k\rangle$ can be discussed in terms of information entropy [4], $S_\alpha = -\sum_k |C_k^\alpha|^2 \ln|C_k^\alpha|^2$. For a given set of single-particle levels, the behavior of entropy in the shell-model basis is mainly determined by the two-body character of interaction, its average strength, and combinatorial growth of the many-body level density [4,20,21]. Therefore the evolution of the delocalization length $\exp(S_\alpha)$ along the spectrum is qualitatively similar for the realistic case and for the random interaction; see Fig. 1 where the comparison is shown with the case (a). In the middle of the spectrum, both entropies are close to the limit $0.48N = 156$ one obtains using the Gaussian orthogonal ensemble (GOE) [4] of random matrices. However, in the realistic case the value of entropy is systematically lower compared to the random interaction. At the same time, the eigenstates of the random interaction are on the GOE limit of complexity with respect to the basis of W eigenstates (inset of Fig. 1). If one calculates the overlap in our model (a) using only the absolute values of the amplitudes (i.e., $[\sum_k | C_k^{\rm gs}(W) C_k^{\rm gs}(R) |]^2$), the result is 0.14, i.e., 7 times larger than the one listed in Table I (0.02). This shows that not only the distribution of amplitudes is important, but their relative signs decide the collectivity of a state.

The lowest states for the random interaction have reduced entropy. In the mixing process of simple configurations, the coherent repulsion creates the energy gap above the ground state which suppresses the further mixing. But the predominantly chaotic nature of these states is confirmed by the weakness of multipole-multipole correlations. As an illustration, results for the reduced quadrupole transition probabilities $B(E2)$ from the lowest 2^+ state to the ground 0^+ state are shown at the end of Table I (units are $e^2\,{\rm fm}^4$) and in Fig. 2. The $B(E2)$ values from the random interactions are divided by the value of 69.5 $e^2\,{\rm fm}^4$ obtained [16] from the (W) interaction. Typically, the $B(E2)$ value is by more than an order of magnitude weaker than obtained with the realistic interactions. Moreover, the maximum $B(E2)$ values out of 1000 samples for each of the four models (a)–(d) (last line in Table I) are smaller than the value obtained for the realistic interaction calculation. One can conclude that this particularly strong collective feature of the realistic interaction has less than 0.1% probability of being reproduced by any of the four random interaction models. In Ref. [9] a *fractional collectivity f* was calculated for a "phonon" operator (selectively maximized for each individual member of the ensemble), and a strong average transition between the first 2^+ state and the 0^+ gs was found: 0.52 ± 0.27. We calculated a similar fraction using everywhere the fixed quadrupole operator,

$$f_{B(E2)} = \frac{|\langle (2^+0)_1 \,|\, E2 \,|\, (0^+0)_{\rm gs}\rangle|^2}{\sum_n |\langle (2^+0)_n \,|\, E2 \,|\, (0^+0)_{\rm gs}\rangle|^2}\,, \qquad (1)$$

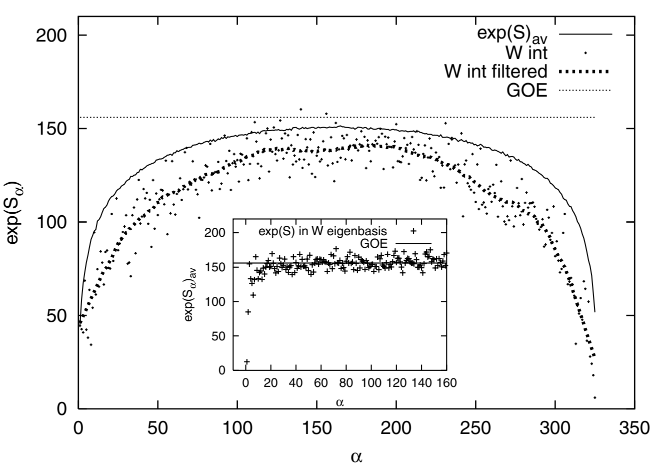

FIG. 1. The average information entropy in the shell-model basis for $J^\pi T = 0^+0$ states over the ensemble (a) of random interactions (solid line), for the realistic interaction W (data, points, and smoothing with a Savitsky-Golay filter [22]), and the GOE limit (horizontal line). The inset shows the entropy of states for the random interaction in the basis of the eigenstates for the W interaction. α represents the eigenstate number ordered by energy.

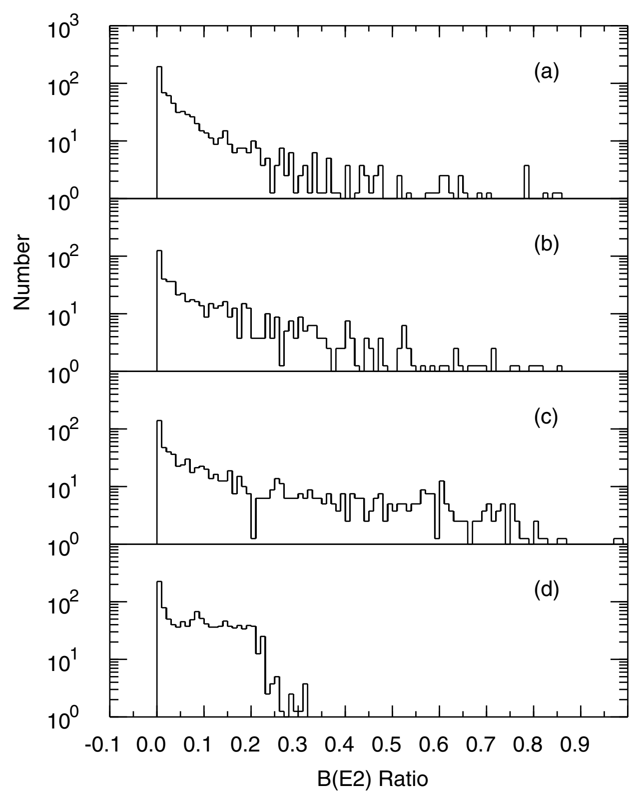

FIG. 2. Distribution of $B(E2)_R/B(E2)_W$ values from the first 2^+0 state to the 0^+0 ground state for models (a)–(d), where $B(E2)_R$ are the values obtained from the random interactions and $B(E2)_W$ is the value from the W interaction.

and we found that in our model (a) its average value is bounded from above by the value of 0.23, significantly smaller than the $f = 0.52$ found in Ref. [9]. Additionally, we found the sum rule in the denominator to be by a factor of 3 smaller for the random ensemble as compared with the realistic case. The question whether the collectivity of the gs, defined in terms of the transition probabilities corresponding to the observable that is known to describe collective effects, such as $B(E2)$, is realistic or not, should be answered negatively. For example, in our model (a) the average value of $B(E2)$ is 7 $e^2\,\mathrm{fm}^4$, the standard deviation is 10.5, (see the $\sigma_{B(E2)}$ line in Table I for the other models), while the realistic value of 69.5 is six standard deviations away from the average value.

The distribution of the $B(E2)$ values for model (a) is close to the Porter-Thomas as one expected for matrix elements of one-body operators between two complicated states [1,18,19]; the 2^+ state is even less ordered than the ground state. A trace of collective strength appears in the model (c). In this respect, one can recall that low-lying collective vibrations, in contrast to high-lying giant resonances that are less sensitive to the residual interactions, emerge only in a superfluid Fermi system. In a normal Fermi system, the low-lying vibrations are not shifted outside the particle-hole continuum and have only a single-particle strength [23]. It means that again we see the pronounced pairing effects only if the residual interaction explicitly contains the pairing part. In model (d), the multipole-multipole correlations generated by the higher components in the pair channel are absent, while the pairing with $J = 0$ alone does not allow for large mixing between the single-particle configurations. The sharp cutoff observed in this model is related to the maximum which can be obtained with pure $j - j$ configurations, e.g., about 19 $e^2\,\mathrm{fm}^4$ for $(d_{5/2})^4$.

In conclusion, with the aid of random rotationally and isospin-invariant two-body interactions in the sd shell model, we have studied the main features of the structure of the ground and low-lying eigenstates as well as complexity of highly excited states. We confirm the strong enhancement of the probability of the quantum numbers $J^\pi T = 0^+0$ for the ground states. However, the resulting ground-state wave functions have only a weak overlap with the realistic ground states that depends on the specific model of randomness. Although the presence of the pairing noticeably increases the percentage of $J^\pi T = 0^+0$ ground states generated by random interactions, its contribution to the collectivity of these states is small. The quadrupole transitions between the lowest 2^+ states and the ground states also do not reveal significant collectivity, in contrast with the results of any realistic interaction. In the eigenbasis of the realistic interaction, the states generated by the random interaction are on the GOE limit of chaoticity. Small hints of coherent components in the low-lying wave functions generated presumably by the off-diagonal pairing matrix elements and observed also in the earlier studies [12] require a more detail analysis. Both random and realistic interactions can generate regular geometric patterns for the low-lying spectra, but it is only the latter that are relevant for those actually found in nuclear physics.

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