Effect of Noise on the Relaxation to an Invariant Probability Measure of Nonhyperbolic Chaotic Attractors

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We study the influence of external noise on the relaxation to an invariant probability measure for two types of nonhyperbolic chaotic attractors, a spiral (or coherent) and a noncoherent one. We find that for the coherent attractor the rate of mixing changes under the influence of noise, although the largest Lyapunov exponent remains almost unchanged. A mechanism of the noise influence on mixing is presented which is associated with the dynamics of the instantaneous phase of chaotic trajectories. This also explains why the noncoherent regime is robust against the presence of external noise.

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Noise is always present in physical and chemical experiments, stronger in biological systems, but also in computer simulations due to roundoff. Its influence on chaotic systems has been explored intensively in the past decades [1–3]. However, it has only been proven for hyperbolic systems that weak noise does not considerably affect their statistical characteristics. This is because hyperbolic systems have several fundamental properties, mainly ergodicity and mixing, which provide the existence of an invariant probability measure of a chaotic attractor [4–6]. However, there are only a few rather artificially constructed hyperbolic attractors and several examples of almost-hyperbolic attractors. For this type of chaotic attractor, the main properties of hyperbolic systems are preserved, except the structural stability [7]. This structural instability does not influence experimentally the observed characteristics of the chaotic behavior. For almost-hyperbolic attractors an invariant probability measure may be introduced as well as for robust hyperbolic sets. Statistical characteristics of almost-hyperbolic attractors are stable to small perturbations and external noise [8].

However, most chaotic systems are nonhyperbolic [7,9]. The problem of the existence of an invariant measure on a nonhyperbolic chaotic attractor involves serious difficulties because it is generally impossible to obtain a stationary probability distribution being independent of an initial distribution. When noise is added to the system, an invariant measure on a nonhyperbolic attractor may also exist [10]. However, in this case characteristics of noisy nonhyperbolic chaos may strongly depend on both the noise statistics and noise intensity [11–14].

The rate of relaxation to an invariant measure is one of the important characteristics of chaotic systems, which is related to the mixing. For axiom-A diffeomorphisms the metric Kolmogorov entropy H_{K} [4,5] defines the rate of mixing. It has been rigorously proven for axiom-A diffeomorphisms that the autocorrelation function decreases exponentially and the correlation time is $\tau_{\mathrm{cor}} = H_{\mathrm{K}}^{-1}$ [15]. Additionally, H_{K} is determined by the sum of positive Lyapunov exponents, i.e., $H_{\mathrm{K}} = \sum_j \lambda_j^{+}$ [16]. However, such exponential estimates are not always valid for flows [17,18]. The problem becomes more difficult for noisy nonhyperbolic chaotic attractors.

In this Letter, we study the rate of relaxation to a stationary probability distribution of nonhyperbolic chaotic attractors of a flow system in the presence of noise. If the noise source is normal and uncorrelated, the temporal evolution of a probability density can be described by the Fokker-Planck equation (FPE). For nonlinear chaotic systems the nonstationary solution of the FPE is difficult enough to find even numerically. Therefore, in this Letter we use the method of stochastic differential equations [14]. By analyzing two different types of nonhyperbolic attractors, a spiral (or coherent) and a noncoherent one [19,20], we find quite different behavior of the relaxation rate. In the coherent case, noise strongly affects the mixing and drastically reduces the correlation time, whereas noise has almost no influence on the noncoherent attractor. It is important to mention that in both cases the maximum Lyapunov exponent remains almost unchanged in the presence of noise. Therefore, there is a strong difference in hyperbolic systems; for the coherent nonhyperbolic system the mixing (i.e., also the predictability) cannot be related simply to the Kolmogorov entropy or the Lyapunov exponents. We explain these findings by means of the dynamics of the instantaneous phase of the trajectories.

For this purpose we investigate the Rössler system which is a prototype of a chaotic nonhyperbolic oscillator [19]. If noise is added to the x component it takes the form

$$\dot{x} = -y - z + \sqrt{2D}\,\xi(t), \qquad \dot{y} = x + ay, \\ \dot{z} = b - mz + xz, \tag{1}$$

where $\xi(t)$ is a normal noise source with the mean value $\langle \xi(t) \rangle \equiv 0$ and correlation $\langle \xi(t)\xi(t+\tau) \rangle \equiv \delta(\tau)$, with $\delta(\cdot)$ being Dirac's function, and D is the noise intensity.

To examine the relaxation to the stationary distribution in this system, we analyze how points situated at an initial

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time in a cube of small size δ around an arbitrary point of the trajectory belonging to an attractor of the system evolve with time. We take $\delta = 0.09$ for the size of this cube and fill it uniformly with $n = 9000$ points. As time goes on, these points in the phase space are distributed throughout the whole attractor. To characterize the convergence to the stationary distribution we follow the temporal evolution of this set of points and calculate the ensemble average $\bar{x}(t) = \frac{1}{n}\sum_{i=1}^{n} x_i(t)$. Because $\bar{x}(t)$ is an oscillating function, we compute the function $\gamma(t_k)$,

$$\gamma(t_k) = |\bar{x}_m(t_{k+1}) - \bar{x}_m(t_k)|, \qquad (2)$$

where $\bar{x}_m(t_k)$ and $\bar{x}_m(t_{k+1})$ are successive extrema of $\bar{x}(t)$. Thus, $\gamma(t_k)$ can be thought of as the amplitude of the mean value oscillations of $\bar{x}$. In Eq. (2) t_k and t_{k+1} are successive time moments corresponding to the extrema of $\bar{x}$. The temporal behavior of $\gamma(t_k)$ allows us to judge the character and the rate of relaxation to the probability measure on the attractor. Besides, we also calculate the largest Lyapunov exponent (LE) λ_1, which is positive in a chaotic regime, and the normalized autocorrelation function $R(\tau)$ of the steady-state oscillations $x(t)$. We define $R(\tau)$ as $R(\tau) = \Psi(\tau)/\Psi(0)$ with $\Psi(\tau) = \langle x(t)x(t + \tau)\rangle - \langle x(t)\rangle^2$, where the angle brackets denote time averaging.

At $a = b = 0.2$, $D = 0$ and, in the parameter m range $[4.25, 9]$, the system (1) has a chaotic attractor which is an example of spiral chaos [19]. The phase trajectory on the spiral attractor rotates with a high regularity around a saddle focus. The autocorrelation function is oscillating and the power spectrum exhibits narrow-band peaks corresponding to the mean rotation frequency, its harmonics, and subharmonics. By virtue of these properties spiral chaos is called coherent. The chaotic attractor of (1) is qualitatively changing as the parameter m increases. For $m > 9$ there occurs a nonhyperbolic attractor of noncoherent type [20], called funnel attractor, which does not already demonstrate regularity in the behavior of the phase trajectory. This regime is referred to as noncoherent chaos.

The calculations performed for $m \in [4.25, 8.5]$ (spiral chaos) and for $m \in [9, 13]$ (noncoherent chaos) allow us to assume that an invariant probability measure exists for the parameter values considered. All the effects being observed for each type of attractor in (1) are qualitatively preserved when the parameter m is varied. In our numeric simulation we fix $m = 6.1$ for the spiral attractor and $m = 13$ for the funnel attractor.

Figure 1 shows the behavior of $\gamma(t_k)$ for both the spiral and the funnel attractor. We find that, in the regime of spiral attractor, noise significantly influences the rate of relaxation to the stationary probability distribution [Fig. 1(a)]. It is strongly decreasing for increasing noise. The decreasing of $\gamma(t_k)$ in time can be approximated as $\sim \exp(-\alpha t)$. The value of α does not depend on the size of the initial cube but significantly increases with increasing D, e.g., $\alpha = 0.02$, 0.03, and 0.09 correspond to curves 1, 2, and 3 in

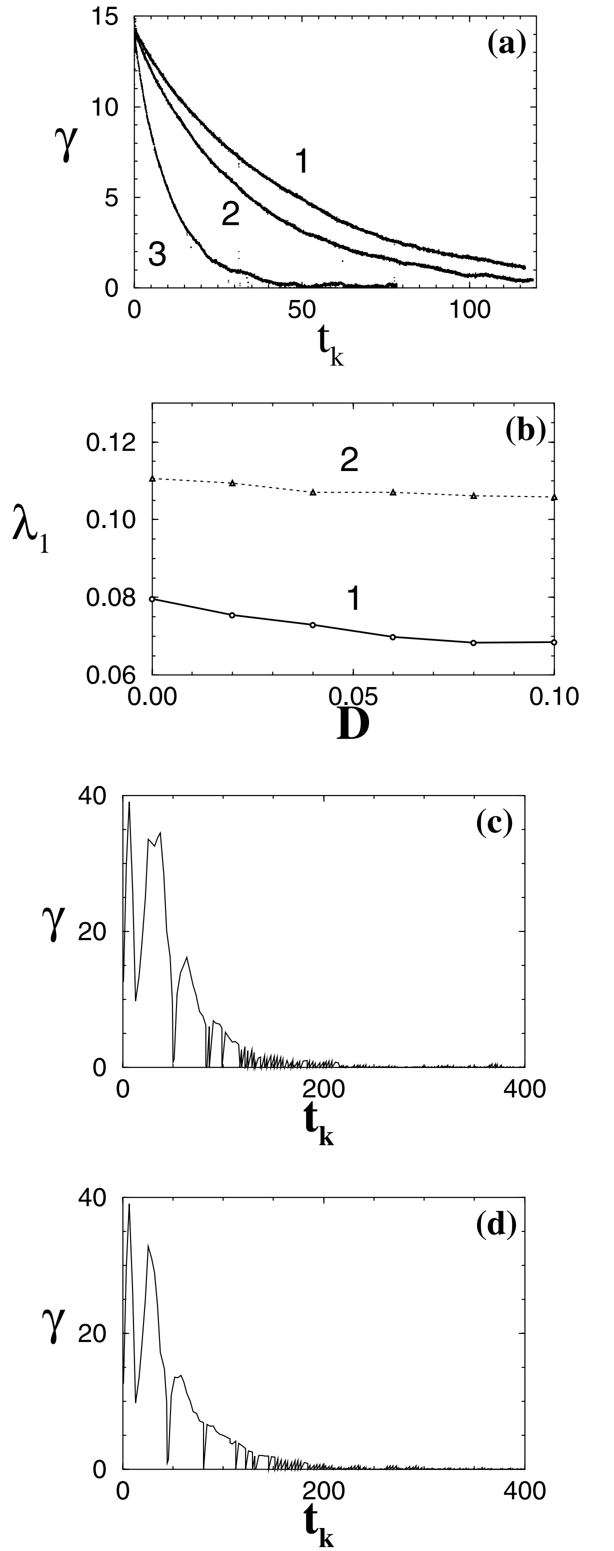

FIG. 1. Characteristics of the rate of mixing for the Rössler system in the regime of spiral ($m = 6.1$) and noncoherent ($m = 13$) chaos. (a) $\gamma(t_k)$ for the spiral attractor (curves 1, 2, and 3 correspond to the noise intensity $D = 0$, $D = 0.001$, and $D = 0.1$, respectively). (b) The largest LE as a function of the noise intensity for the spiral (curve 1) and the funnel (curve 2) attractor. Function $\gamma(t_k)$ in the regime of noncoherent chaos without noise (c) and in the presence of noise with $D = 0.01$ (d).

Fig. 1(a), respectively. However, contrary to this strong influence on noise, the largest LE λ_1 [Fig. 1(b)] decreases only slightly as the noise intensity is varied in the interval $0 \leq D \leq 0.1$. We find a quite different situation for the funnel attractor. There the rate of relaxation is practically insensitive to noise perturbations [Figs. 1(c) and 1(d)]; no considerable changes are observed in the behavior of $\gamma(t_k)$ when noise sources are added to the system. As before, the positive LE changes weakly with increasing noise intensity [see Fig. 1(b), curve 2]. Next we analyze these different types of responses to noise in more detail.

It is well known that noncoherent chaos exhibits a close similarity to random processes. This fact can be verified, e.g., by means of the autocorrelation function $R(\tau)$ for the spiral and the funnel attractor in system (1) (Fig. 2). Our numerical experiments show that the correlation times $\tau_{\rm cor}$ are essentially different for these two chaotic regimes: $\tau_{\rm cor} \approx 9500$ for the spiral chaos ($m = 6.1$) and $\tau_{\rm cor} \approx 40$ for the funnel attractor ($m = 13$). (The correlation time is usually understood as the time in the course of which the autocorrelation function decreases in e times.) On the one hand, in the case of coherent chaos the correlation time decreases dramatically in the presence of noise [Fig. 2(a)], e.g., $\tau_{\rm cor} \approx 5500$ for $D = 0.001$ and $\tau_{\rm cor} \approx 200$ for $D = 0.1$. On the other hand, the envelopes of the autocorrelation functions for the funnel attractor practically coincide in the deterministic case with those in the presence of noise [Fig. 2(b)]. Hence, noncoherent chaos, which is nonhyperbolic, demonstrates some property of hyperbolic chaos; i.e., "dynamical stochasticity" turns out to be much stronger than that imposed from an external (additive) one [5]. It is also worth noting another important finding of our simulations. In the regime of spiral chaos the rate of mixing is not uniquely determined by the largest LE but depends strongly on the noise intensity. These data are interesting and require more detailed consideration.

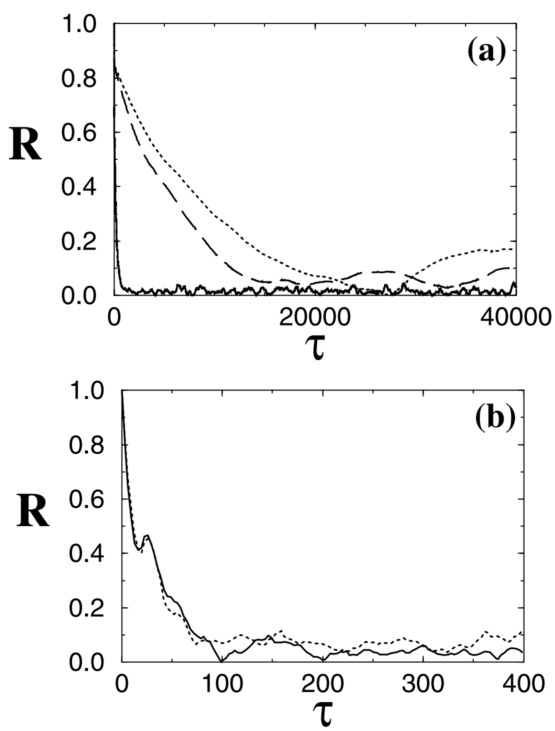

FIG. 2. Envelopes of autocorrelation functions for system (1): (a) in the regime of spiral chaos for $m = 6.1$ and at $D = 0$ (dotted line), $D = 0.001$ (dashed line), and $D = 0.1$ (solid line); (b) in the regime of noncoherent chaos for $m = 13$ and at $D = 0$ (dotted line), and $D = 0.01$ (solid line).

We have found that the largest LE is insensitive to fluctuations, whereas in certain cases the correlation time changes considerably under the influence of noise. These facts testify that the Kolmogorov entropy is not the unique characteristic responsible for the mechanism of mixing on a nonhyperbolic attractor. We suppose that the essential effect of noise on relaxation to the stationary distribution may be associated with peculiarities of the system dynamics. Since the trajectory rotates almost regularly on the spiral attractor, the relaxation process appears to be very long. The addition of noise to the system destroys the relative regularity of the trajectory and, consequently, the rate of mixing significantly increases.

Numerical calculations performed for chaotic model return maps have shown that the rate of mixing in a map is defined by the positive LE and depends weakly on the noise level. It remains to be seen how noise influences a regularity of intersections of phase trajectories with some secant planes. Therefore, we examine a set of trajectories which start from nearby attractor points of the differential system (1) and measure the time moments of their intersection with a certain secant plane.

It is known that, for chaotic oscillators, one can introduce the notions of instantaneous amplitude and phase [21]. For the Rössler system (1) the instantaneous phase can be written as follows

$$\Phi(t) = \arctan\left(\frac{y(t)}{x(t)}\right) + \pi N(t), \qquad (3)$$

where $N(t) = 0, 1, 2, \ldots$ is the number of intersections of the trajectory with the plane $x = 0$.

We consider the instantaneous phase difference $\Delta_n = \Phi_2(t_n) - \Phi_1(t_n)$ for two initially close trajectories of system (1) at the time t_n when the first trajectory crosses the plane $x = 0$. Thus, we obtain a sequence of Δ_n for the spiral as well as the funnel attractor. We again find that, in the regime of spiral chaos [Fig. 3(a)], noise drastically changes the temporal behavior of the phase differences of two initially neighboring trajectories. When $D = 0$, the phase difference varies smoothly and slowly, with the exception of fine-scaled changes within $\pm\pi$. However, the addition of noise leads to changes larger than 2π in short time intervals. Thus, mixing is strongly enhanced under the influence of noise. It is important to emphasize that phase changes are very typical for the noncoherent attractor already in a purely deterministic case. Therefore, the

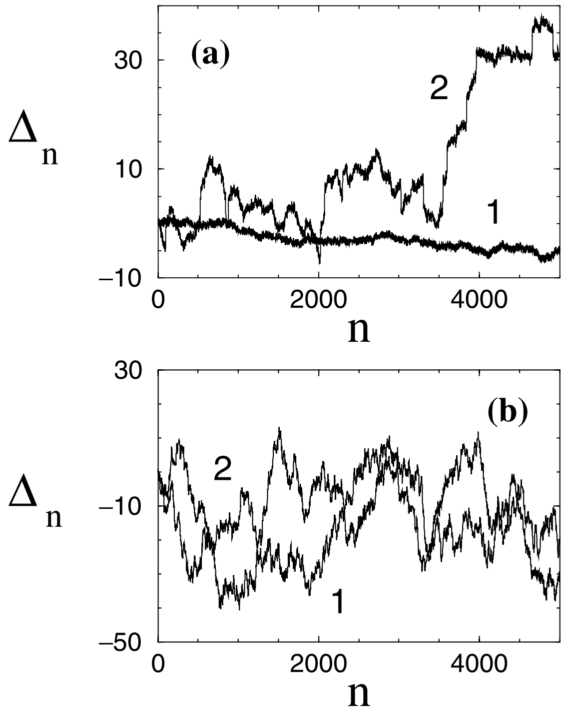

FIG. 3. Instantaneous phase difference Δ_n on n for $m = 6.1$ (a) and $m = 13$ (b) in the noise-free case (curves 1) and in the presence of noise with intensity $D = 0.1$ (curves 2).

variations of Δ_n are qualitatively the same without and with the presence of noise [see Fig. 3(b)]. Thus, for the funnel attractor the mixing process is practically similar both in the noisy and in the noise-free case.

In conclusion, we have shown that both the time of relaxation to a stationary distribution and the rate of mixing on noisy nonhyperbolic attractors of flow systems are determined not only by the positive LE but also by the noise intensity and the instantaneous phase dynamics of chaotic oscillations. This property has a strong impact on evaluating the predictability of such systems and on modeling them. It is, therefore, very important to apply the proposed concept to the study of complex systems in various fields ranging from economics to living systems, especially in neuroscience.

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