Quantum Hall Plateau Transition at Order $1/N$

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The localization behavior of noninteracting two-dimensional electrons in a random potential and strong magnetic field is of fundamental interest for the physics of the quantum Hall effect. In order to understand the emergence of power-law delocalization near the discrete extended-state energies $E_n = \hbar\omega_c(n + \frac{1}{2})$, we study a generalization of the disorder-averaged Liouvillian framework for the lowest Landau level to N flavors of electron densities ($N = 1$ for the physical case). We find analytically the large-N limit and $1/N$ corrections for all disorder strengths: at $N = \infty$ this gives an estimate of the critical conductivity, and at order $1/N$ an estimate of the localization exponent ν.

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The explanation by Laughlin [1] of the integer quantum Hall effect depends upon an understanding of the localization of electrons by disorder in a strong magnetic field. Without disorder, the single-electron eigenstates fall into Landau levels at isolated energies $E_n = \hbar\omega_c(n + \frac{1}{2})$, $n = 0, 1, 2, \ldots$, separated by the cyclotron energy $\hbar\omega_c$. The effect of a weak disorder potential is to displace some weight from the δ-function peaks at E_n into localized states at nearby energies. Extended states persist at energies E_n, and the localization length $\xi(E)$ of states at energy E diverges as $E \to E_n$ according to a power law $\xi(E) \propto (E - E_n)^{-\nu}$. Experimental results on disordered samples [2] are consistent with the value $\nu \approx 2.35 \pm 0.05$ obtained from numerical calculations [3,4] on the lowest Landau level (LLL).

Current belief is that the quantum Hall plateau transition lies in an entirely different universality class from the zero-field case, characterized by "two-parameter scaling" [5,6] and a topological term in the σ-model description [6]. In particular, there is now an understanding of the minimal features required to obtain numerically scaling behavior near the transition [4], which is in some respects similar to classical percolation, but with different universal properties as a result of quantum tunneling and interference [7]. There is also some understanding of the apparent insensitivity to interactions of some critical indices such as ν [8]. However, relatively little progress has been made in finding an analytically tractable description of the localization properties near the critical energy. The subject of this paper is a generalization of the problem to multiple flavors of electron densities, allowing an analytical approach to the transition. Our discussion is based on the Liouvillian approach [9], reviewed below.

The generalization of the disorder-averaged action to N flavors of electron densities gives a simple mean-field-like theory in the large-N limit. Other large-N approaches such as [10] typically generalize the noninteracting problem before the disorder average and do not obtain ν. At first order in the small parameter $1/N$, we recover anomalous scaling of the localization length, i.e., a value for the critical exponent ν. Thus we find an analytically tractable quantum description whose physics seems to connect smoothly in the parameter N to the plateau transition ($N = 1$). The control parameter $1/N$ allows a systematic expansion around $N = \infty$, and the large-N limit and $1/N$ corrections can be found analytically for all disorder strengths.

The localization properties of electrons at energy E are contained in the correlation function of the LLL-restricted density operators $\bar{\rho}_{\mathbf{q}}$:

$$\tilde{\Pi}(q, t; E) = \frac{-i\Theta(t)}{N_L \hbar \ell^2} \langle\langle \mathrm{Tr}\bar{\rho}_{\mathbf{q}}(t)\bar{\rho}_{-\mathbf{q}}(0)\delta(E - H)\rangle\rangle . \quad (1)$$

Here N_L is the number of states in the LLL ($N_L \to \infty$ taken below), $\ell = \sqrt{\hbar c/eB}$, and $\langle\langle\ \rangle\rangle$ indicates the quenched disorder average.

The Liouvillian approach uses the integral over E of $\tilde{\Pi}(q, t; E)$:

$$\tilde{\Pi}(q, t) \equiv \int dE\, \tilde{\Pi}(q, t; E) = \frac{-i\Theta(t)}{N_L \hbar \ell^2} \langle\langle \mathrm{Tr}\bar{\rho}_{\mathbf{q}}(t)\bar{\rho}_{-\mathbf{q}}(0)\rangle\rangle . \quad (2)$$

The key to the approach is that $\tilde{\Pi}(q, t)$ still contains information about electron localization but is more easily calculated than the fixed-energy quantity $\tilde{\Pi}(q, t; E)$. The disorder average for the Fourier transform $\tilde{\Pi}(q, \omega)$ was carried out numerically in [9] and shown to obey the form $\omega\, \mathrm{Im}\tilde{\Pi}(q, \omega) = \omega^{1/2\nu} f(q^2/\omega)$, where f is an unknown scaling function. At mean-field level [9], only diffusion is found: our model gives a systematic expansion beyond this mean-field result. A major goal of this paper is to obtain the prefactor $\omega^{1/2\nu}$ by a $1/N$ expansion of $\tilde{\Pi}(q, \omega)$.

The scaling form for $\tilde{\Pi}(q,\omega)$ comes about because at large R, only states with energies satisfying

$$\frac{|E - E_c|}{E_c} < \left(\frac{a_d}{R}\right)^{1/\nu} \tag{3}$$

will have $\xi(E) > R$ and hence contribute to the correlation function at long enough times. Here a_d is a nonuniversal length set by the disorder potential. Thus the integral over energy which gives $\tilde{\Pi}(q,\omega)$ is only nonzero in a window of size proportional to $q^{1/\nu}$. For states delocalized on the scale $R = 1/q$, $\tilde{\Pi}(q,\omega;E)$ follows ordinary diffusive scaling ($\omega\,\mathrm{Im}\tilde{\Pi}$ is a function of q^2/ω). A prefactor $\omega^{1/2\nu}$ rather than $q^{1/\nu}$ gives the scaling function a simple form: for $q^2 \ll \omega$

$$\omega\,\mathrm{Im}\tilde{\Pi}(q,\omega) \approx D(\omega)(q^2/\omega), \tag{4}$$

with the frequency-dependent diffusion constant $D(\omega) = D_0\omega^{1/2\nu}$. This form also applies in the classical percolation limit studied by Gurarie and one of us [11].

The LLL-projected density operators $\bar{\rho}_{\mathbf{q}}$ are related to the operators $\tau_{\mathbf{q}}$ of the magnetic translation group through [12] $\bar{\rho}_{\mathbf{q}} = e^{-(1/4)\ell^2 q^2}\tau_{\mathbf{q}}$. The noninteracting LLL-projected Hamiltonian is $H = \sum_{\mathbf{q}} v(-q)\bar{\rho}_{\mathbf{q}}$, with v the Fourier-transformed random potential. Then using the commutation relation for the operators $\tau_{\mathbf{q}}$

$$[\tau_{\mathbf{q}}, \tau_{\mathbf{r}}] = 2i \sin\left(\frac{\ell^2}{2}\,\mathbf{q} \wedge \mathbf{r}\right)\tau_{\mathbf{q}+r}, \tag{5}$$

we obtain the evolution equation for the magnetic translation operators:

$$\dot{\tau}_{\mathbf{q}} = -i \sum_{\mathbf{q}'} \mathcal{G}_{\mathbf{q}\mathbf{q}'}\tau_{\mathbf{q}'}, \tag{6}$$

in terms of the Girvian

$$\mathcal{G}_{\mathbf{q}\mathbf{q}'} = \frac{2i}{\hbar}\, v(\mathbf{q} - \mathbf{q}')e^{-(1/4)\ell^2|\mathbf{q}'-\mathbf{q}|^2} \sin\left(\frac{\ell^2}{2}\,\mathbf{q}' \wedge \mathbf{q}\right). \tag{7}$$

Then $\tilde{\Pi}(q,\omega)$ is a one-body correlation function of the Girvian:

$$\tilde{\Pi}(q,\omega) = \frac{1}{\hbar\ell^2}\langle\langle\langle \mathbf{q}|(\omega - \mathcal{G})^{-1}|\mathbf{q}\rangle\rangle\rangle. \tag{8}$$

Here the states $|\mathbf{q}\rangle$ and operator $\mathcal{G}$ are defined through $\langle\mathbf{q}|\mathcal{G}|\mathbf{q}'\rangle = \mathcal{G}_{\mathbf{q}\mathbf{q}'}$.

Now we take the continuum limit $N_L \to \infty$ and assume a white-noise disorder potential $\langle\langle v(\mathbf{q})v(-\mathbf{q})\rangle\rangle = \frac{2\pi v^2}{L^2}\delta(\mathbf{q} + \mathbf{q}')$. The physical frequency ω_0 is replaced by the dimensionless combination $\omega = h\omega_0\ell/v$. The physical propagator is $\tilde{\Pi}(q,\omega_0) = \frac{h\ell}{v}\Pi(q,\omega)$, with the dimensionless propagator $\Pi(q,\omega) \to \frac{1}{\omega}$ in the clean limit $\omega \to \infty$. All momenta are dimensionless (scaled by magnetic length ℓ) in the following.

The result of disorder averaging is an interacting theory which can be expressed through a functional integral over both bosonic ϕ and Grassmann ψ variables [9]:

$$\Pi(q,\omega) = -i\int D\bar{\phi}\,D\phi\int D\bar{\psi}\,D\psi\;\bar{\phi}_q\phi_q e^{-F(\omega)},$$

$$F(\omega) = -i\omega\int d\mathbf{q}\,(\bar{\phi}_{\mathbf{q}}\phi_{\mathbf{q}} + \bar{\psi}_{\mathbf{q}}\psi_{\mathbf{q}}) + \int_{1,2,3,4} f(1,2,3,4)[\bar{\phi}_{\mathbf{q}1}\bar{\phi}_{\mathbf{q}2}\phi_{\mathbf{q}3}\phi_{\mathbf{q}4} + 2\bar{\psi}_{\mathbf{q}1}\bar{\phi}_{\mathbf{q}2}\phi_{\mathbf{q}3}\psi_{\mathbf{q}4} + \bar{\psi}_{\mathbf{q}1}\bar{\psi}_{\mathbf{q}2}\psi_{\mathbf{q}3}\psi_{\mathbf{q}4}]. \tag{9}$$

The effective interaction from disorder averaging is

$$f(1,2,3,4) = \frac{1}{\pi}\,e^{-(1/2)|\mathbf{q}_1-\mathbf{q}_4|^2}\delta(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4)\sin\left(\frac{1}{2}\,\mathbf{q}_1 \wedge \mathbf{q}_4\right)\sin\left(\frac{1}{2}\,\mathbf{q}_2 \wedge \mathbf{q}_3\right). \tag{10}$$

The effect of the additional Grassmann variables ("supersymmetry") is simply to eliminate processes with more than one density line. We develop a natural generalization of this interacting problem to N flavors of densities ($N = 1$ is the original problem). The $N \to \infty$ limit gives an approximate propagator which is similar to the self-consistent Born approximation of [9]. The utility of the large-N approach is that it gives a systematic expansion in the parameter $1/N$ around the diffusive $N \to \infty$ result, while at the point $N = 1$ the theory describes the exact localization properties of the plateau transition.

The N-flavor generalization of the Lagrangian density is (only the bosonic part is given, for compactness, and the flavor indices i, j run from 1 to N)

$$\mathcal{L}_N = -i\omega\bar{\phi}^i_q\phi^i_q + \frac{f(1,2,3,4)}{N}[c_1\bar{\phi}^i_{q_1}\bar{\phi}^j_{q_2}\phi^i_{q_3}\phi^j_{q_4} + c_2\bar{\phi}^i_{q_1}\bar{\phi}^i_{q_2}\phi^j_{q_3}\phi^j_{q_4} + c_3\bar{\phi}^i_{q_1}\bar{\phi}^j_{q_2}\phi^j_{q_3}\phi^i_{q_4}]. \tag{11}$$

The real coefficients c_i in (11) reproduce the correct $N \to 1$ limit provided that $c_1 + c_2 + c_3 = 1$, so there is a two-parameter family of generalizations. The vertex with coefficient c_3 in (11) does not contribute as $N \to \infty$ and does not seem to affect scaling qualitatively at order $1/N$, so it is dropped for simplicity. [13] We specialize to $c_1 = c_2 = 1/2$ in what follows: the choice of these coefficients equal is "natural" in that the classes of diagrams selected by the two vertices have equal weight at order $1/N$ with $N = 1$, as they do in the full theory (i.e., all orders in $1/N$) at $N = 1$. For generic c_i the theory has a $\mathrm{U}(1\,|\,1) \times \mathrm{SO}(N)$ symmetry, which at the point $c_3 = 1$ (N decoupled systems) becomes $\mathrm{U}(N\,|\,N)$.

In the $N \to \infty$ limit, the diagrams with k interaction lines which contribute to $\langle \bar{\phi}_q^1 \phi_q^1 \rangle$ are the diagrams where no interaction lines cross, which have the maximum k free choices of flavor index (i.e., degeneracy N^k). Only the first interaction term of (11) affects this limit. The noncrossing propagator sums these diagrams and satisfies an integral equation depicted graphically in Fig. 1:

$$\Pi_B(q,\omega) = \frac{1}{\omega} + \frac{\Pi_B(q,\omega)}{\pi\omega} \int d\mathbf{q}' \sin^2\left(\frac{1}{2}\,\mathbf{q} \wedge \mathbf{q}'\right) e^{-(1/2)|\mathbf{q}-\mathbf{q}'|^2} \Pi_B(q',\omega)\,. \tag{12}$$

The notation $\Pi_B(q,\omega)$ is that of [9]. In the limit $q, \omega \to 0$, $q^2 \ll \omega$, the noncrossing result $\Pi_B(q,\omega)$ shows diffusive behavior ($\omega\,\mathrm{Im}\Pi = D_0 q^2/\omega$) without the prefactor $\omega^{1/2\nu}$.

The zeroth-order diffusion constant D_0 is a factor of $\sqrt{2}$ smaller here than in the self-consistent Born approximation of [9] (the calculation there corresponds to the choice $c_1 = 1$). The resulting estimate of the critical conductivity at the transition, obtained from D_0 and the exact density of states [14] through the Einstein relation, is $\sigma_{xx} \approx 0.614\frac{e^2}{h}$ compared to the numerical result $\sigma_{xx} = (0.54 \pm 0.04)\frac{e^2}{h}$ [15] and $\sigma_{xx} = [(\sqrt{3}\,e^2)/(4h)] \approx 0.433\frac{e^2}{h}$ of [11]. Although the large-N estimates of σ_{xx} from the Liouvillian are of the right order, it should be noted that (unlike ν) a universal σ_{xx} has not yet been found from $\Pi(q,\omega)$ obtained using numerical diagonalization.

The noncrossing propagator has simple behavior at large q, where the argument of the integrand in (12) is rapidly oscillatory. In this limit

$$\Pi_B(\infty,\omega) = \frac{1}{\omega} + \frac{\Pi_B(\infty,\omega)^2}{\omega}\,, \tag{13}$$

or $\Pi_B(\infty,\omega) \approx -i + \frac{\omega}{2}$ for small ω.

The first corrections to the noncrossing propagator have $k-1$ free choices of index for a diagram of k lines. The corrections consist of all maximally crossed (MC) diagrams of two or more lines (using the noncrossing propagator) (Fig. 2a), and in addition possible "rainbows" over the maximally crossed portion (Fig. 2b). The sum of all maximally crossed diagrams can be obtained from the ladder sum represented in Fig. 3 since maximally crossed diagrams are related to ladder diagrams after cutting the center propagator line and pivoting.

The sum of all ladder diagrams with incoming momenta $\mathbf{q}_1, \mathbf{q}_2$ and momentum transfer $\mathbf{l}$ as labeled in Fig. 3, denoted by $V(\mathbf{q}_1, \mathbf{q}_2; \mathbf{l})$, satisfies the integral equation

$$V(\mathbf{q}_1,\mathbf{q}_2;\mathbf{l}) = V_0(\mathbf{q}_1,\mathbf{q}_2;\mathbf{l}) + \int \frac{d\mathbf{l}'}{\pi}[V_0(\mathbf{q}_1 + \mathbf{l}', \mathbf{q}_2 - \mathbf{l}'; \mathbf{l} - \mathbf{l}')\Pi_B(q_1 + l', \omega)\Pi_B(q_2 - l', \omega)V(\mathbf{q}_1,\mathbf{q}_2;\mathbf{l}')], \tag{14}$$

where V_0 is the original interaction

$$V_0(\mathbf{q}_1,\mathbf{q}_2;\mathbf{l}) \equiv \frac{e^{-(1/2)|\mathbf{l}|^2}}{\pi} \sin\left(\frac{\mathbf{q}_1 \wedge \mathbf{l}}{2}\right) \sin\left(\frac{\mathbf{q}_2 \wedge \mathbf{l}}{2}\right). \tag{15}$$

Note that V has an implicit ω dependence. The content of (14), discussed below, is that V has a diffusion pole when $\mathbf{q}_2 \approx -\mathbf{q}_1$:

$$V(\mathbf{q},\mathbf{q}';\mathbf{q}' - \mathbf{q}) \propto \frac{1}{i\omega + D_1(q + q')^2}\,. \tag{16}$$

The physics of (16) is similar to that of the weak-localization (WL) logarithmic singularity in two dimensions [16], but with two major differences: the disorder-generated effective interaction involves four bosonic density operators and hence eight rather than four fermionic operators, and time-reversal symmetry is broken by the magnetic field. The singularity results when the integral equation (14) becomes nearly $V = V_0 + V$, i.e., when the integral operator on the right-hand side has an eigenvalue going to 1. Consider a rung of a long ladder diagram. The intermediate propagator momenta on the nth rung from an end of the ladder have magnitude proportional to $\sqrt{n}$ (momenta walk randomly in the plane) so most momenta in a long ladder can be assumed large. Each rung adds two propagators $\Pi_B(\infty,\omega) \approx -i + \omega/2$, and the interaction averages to $-1 + \frac{1}{8}(q + q')^2$. This gives the estimate $D_1 = \frac{1}{8}$ in (16).

The above derivation assumes white-noise disorder in the Girvian, although physical white-noise disorder corresponds to disorder of correlation length $\sim\ell$ in $\mathcal{G}$ [9].

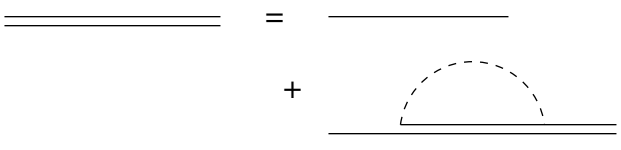

FIG. 1. Diagrammatic representation of the large-N propagator equation (13). The double line is the noncrossing propagator.

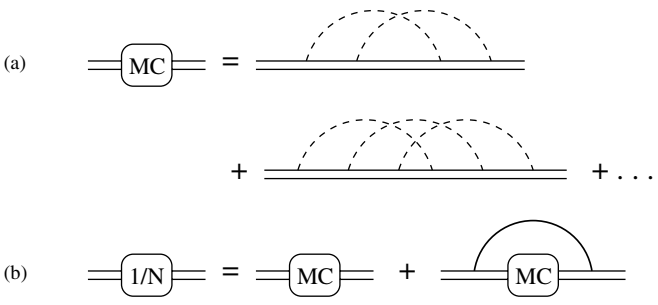

FIG. 2. Diagrams contributing to $1/N$ propagator correction. The solid interaction line in (b) indicates one or more noncrossing interaction lines. Each diagram has one fewer free index choice than the noncrossing diagrams of the same order.

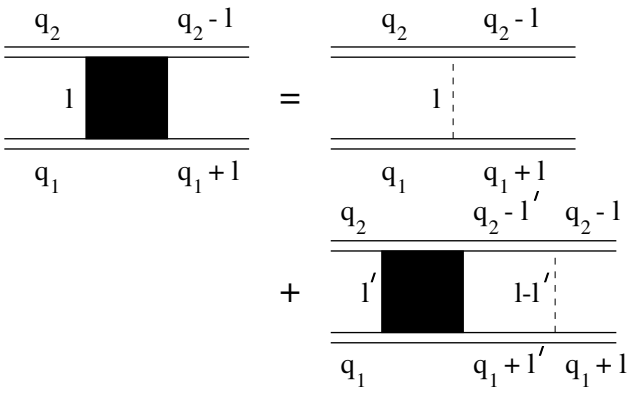

FIG. 3. Schematic representation of the sum of ladder diagrams. The solid block is defined as the sum of one or more rungs with the specified total momentum transfer.

Another assumption is that the lack of an upper momentum cutoff in the Liouvillian approach does not soften the singularity (16) to a logarithm. The singularity is found numerically to persist without these assumptions, suggesting that the high-momentum degrees of freedom do not change the scaling behavior qualitatively, as in [17]. We call the limit with point disorder and finite cutoff the WL limit as these assumptions are standard in that context.

The contribution of all maximally crossed diagrams to the propagator is

$$\frac{\Pi_{\mathrm{MC}}(q,\omega)}{\Pi_B(q,\omega)^2} = \int d\mathbf{q}'\,\Pi_B(q',\omega)(V - V_0) \times (\mathbf{q},\mathbf{q}';\mathbf{q}' - \mathbf{q}). \tag{17}$$

Here the subtraction of V_0 removes the first ladder diagram, which has no crossings and is already in Π_B. Integrating q' in (16) leads to a $\log\omega$ contribution in Π_{MC}.

The sum of maximally crossed diagrams (17) already shows nondiffusive scaling (an anomalous prefactor in ω), which is modified quantitatively but not qualitatively by adding rainbows over the MC diagrams. There is a logarithmic divergence in the sum of maximally crossed diagrams, leading to a diffusion constant

$$D(\omega) = D_0 + \frac{a\log\omega}{N} + O(N^{-2}) \approx D_0 \exp\left(\frac{a\log\omega}{ND_0}\right) = D_0\omega^{a/ND_0} \tag{18}$$

using the standard device of reexponentiation in large-N theories to estimate critical exponents. Hence

$$\frac{1}{2\nu} = \frac{a}{ND_0} + O(N^{-2}). \tag{19}$$

The coefficient a is found from the numerical solution of (14) and (17). Figure 4 shows sample data from this calculation. Numerically $D_0 = 0.682$, $a = 0.18 \pm 0.02$, and therefore $\nu = (1.89 \pm 0.1)N$. In the WL limit the addition of overloops (Fig. 2b) is found analytically to reduce a and increase ν by a factor of 2. Preliminary numerical results [13] are that the correction is in the same direction

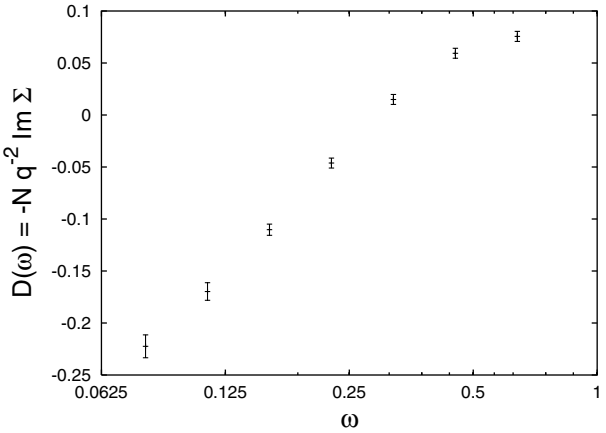

FIG. 4. Numerical results for maximally crossed contribution to diffusion constant $D(\omega)$, on a semilog scale. Here $\Sigma = \omega - 1/\Pi$.

for the full model. Although this paper has focused on the LLL plateau transition, the large-N Liouvillian approach may also be useful for other noninteracting quantum Hall transitions with a discrete spectrum of extended states.

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