Detecting Phase Synchronization in a Chaotic Laser Array

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Detection of phase synchronization of coupled chaotic oscillators is examined experimentally for the case of a linear laser array. Phase variables are computed by applying a Gaussian filter, peaked at a positive frequency, to the signal obtained from the intensity time series of the individual lasers. Relationships between different frequency components of the oscillator dynamics that are not otherwise apparent are unambiguously detected.

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Synchronization of chaotic systems has been explored extensively over the past decade; for a review, see Ref. [1]. Various definitions of synchronization have evolved. The concept of generalized synchronization [2] has been introduced in addition to the obvious notion of identical synchronization, and measures of the extent of synchronization have been defined. Recently another concept in synchronization, namely, phase synchronization, has been shown to be a useful tool for the analysis of chaotic signals arising in a variety of situations. Phase synchronism of chaos was originally pointed out by Pikovsky [3] and by Stone [4] for the case in which a chaotic oscillator is perturbed by an external periodic signal (we call this "periodic pacing"). More recently, the idea of phase synchronization between coupled chaotic oscillators has been introduced [5]. This concept has proven to be very useful, especially in the analysis of biological data (e.g., Ref. [6]). Experimental observation of phase synchronization in chaotic physical systems has been recently reported [7–10]. For a review of research in phase synchronization of chaos, see Ref. [11].

In this Letter we focus on the detection of phase synchronization in a chaotic coupled laser system. The intensity time series studied are nonstationary, noisy, and of limited duration. This situation is common in many experimental settings. The introduction of a suitable quantitative definition for the phase of a chaotic signal is the key idea in any study of phase synchronization [6]. It has been found that, for time series generated by chaotic computer models, such as the Rössler system, the means by which phase is computed is not critical; various reasonable definitions appear to be equally effective in evidencing (or not evidencing) phase synchronism. In contrast, for our experimental data set we find that the definition of phase can be crucial to the detection of phase synchronization. We demonstrate a method that allows us to detect and quantify phase synchronization for our data.

In particular, we report here the observation of phase synchronization in a linear laser array consisting of three elements with chaotic intensity fluctuations. Arrays of coupled lasers often display chaotic dynamics [12–15]; individual intensity time series for the lasers are complex, but they may display synchronization in dramatic ways. This is illustrated by the identical intensity synchronization observed between the two outer lasers in a three laser linear array [15] where the lasers are nearest-neighbor coupled. The center laser is not synchronized in any obvious way to the outer ones, though it mediates the interaction and synchronization of the outer elements. While it is simple to identify identical synchronization of intensities, it is much more difficult to discern relationships between the dynamics of elements of the array that are not identically synchronized. Typical experimental measurements record intensity time series without any determination of the phases of the fields for the laser elements. The relative *optical* phases of the array elements determine the intensity distributions in the far field. Phase dynamics and amplitude chaos were studied experimentally and numerically for two spatially coupled lasers [12,13], where phase locking was determined through measurements of interference fringe visibilities. It is difficult, however, to acquire and analyze dynamically evolving interference patterns at time scales comparable to those for the fluctuations of the fields (microseconds or shorter). Therefore, it is important to define new, experimentally accessible phase variables that will allow quantitative detection of phase synchronization between the array elements.

We consider now a fairly general means by which a phase may be associated with a real scalar signal, $I(t)$. Representing $I(t)$ by its Fourier transform $u(\nu)$, $I(t) = (2\pi)^{-1}\int_{-\infty}^{\infty}\exp(i\nu t)u(\nu)\,d\nu$, we note that the time variation of each Fourier component, $u(\nu)e^{i\nu t}$, is a complex number whose phase continually increases (decreases) with time for $\nu > 0$ ($\nu < 0$). Thus, one way to introduce a phase is to suppress the negative ν components by replacing $u(\nu)$ by $2\theta(\nu)u(\nu)$ [where $\theta(\nu)$ is the unit step function, $\theta(\nu) = 1$ for $\nu > 0$ and $\theta(\nu) = 0$ for $\nu < 0$]. In this case, we obtain a superposition of rotating complex numbers all of which have increasing phase,

$$V_A(t) = \pi^{-1}\int_0^{\infty} e^{i\nu t}u(\nu)\,d\nu\,. \quad (1)$$

Thus we may reasonably expect $V_A(t)$ to execute rotation in the complex plane with continually increasing phase. The function $V_A(t)$ is Gabor's "analytic signal" [16], which has been recently introduced for the purpose of the study of phase synchronization of chaos in Ref. [5]. Noting that the inverse transform of $2\theta(\nu)$ is $\delta(t) + \frac{i}{\pi} P \frac{1}{t}$, we can express the analytic signal as

$$V_A(t) = I(t) + iI_H(t) = I(t) + \frac{i}{\pi} I(t) \odot P\,\frac{1}{t}, \quad (2)$$

where $I_H(t)$ and $I(t)$ are related by the Hilbert transform $I_H(t) = \pi^{-1} P \int_{-\infty}^{\infty} dt'\, I(t')/(t - t')$, $\odot$ denotes convolution, and $P\frac{1}{t}$ is the principal part of $1/t$. Writing

$$V_A(t) - \langle V_A \rangle = R_A(t) e^{i\Phi_A(t)}, \quad (3)$$

where $R_A(t)$ and $\Phi_A(t)$ are real, and $\langle V_A \rangle$ is the time average of $V_A(t)$, we call $\Phi_A(t)$ the *analytic phase*. Here we note that it is useful to consider more general choices. In particular, we can replace $u(\nu)$ in the original Fourier transform by $f(\nu)u(\nu)$, where $f(\nu)$ is suitably chosen [Eq. (2) corresponds to $f(\nu) = 2\theta(\nu)$]. Specifically, we will be interested in the choice of a Gaussian for $f(\nu)$, $f(\nu) = \exp[-(\nu - \nu_o)^2/2\sigma^2]$; this gives

$$V_G(t) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu\, e^{i\nu t} e^{-(\nu-\nu_o)^2/(2\sigma^2)} u(\nu) = I(t) \odot F(t), \quad (4)$$

where $F(t) = \sigma(\sqrt{2\pi})^{-1} \exp[-i\nu_o t + \sigma^2 t^2/2]$. We then define a Gaussian phase by

$$V_G(t) - \langle V_G \rangle = R_G(t) e^{i\Phi_G(t)}. \quad (5)$$

The frequency ν_o and the Gaussian's width σ are parameters in the definition of $\Phi_G(t)$. Note that, similar to the choice $f(\nu) = 2\theta(\nu)$ resulting in the analytic signal, the choice of a Gaussian again emphasizes positive frequencies (we take $\nu_o > 0$). We find that application of a frequency bandpass filter as in Ref. [6] produces similar results for our data. For our experimental data set, we have tested for phase synchronization using the phase definitions $\Phi_A(t)$ and $\Phi_G(t)$, as well as some others [17]. We find that the Gaussian phase is superior for our purposes, and we believe that this may also be the case in other situations involving nonstationary, noisy time series of limited duration [18].

The chaotic system studied consists of three parallel, laterally coupled neodymium doped yttrium aluminum garnet (Nd:YAG) lasers ($\lambda = 1064$ nm) of approximately equal average intensities [15]. A thick intracavity étalon ensures single longitudinal mode operation. Coupling through the electric fields of the individual beams exists only for adjacent pairs. This system of nearest-neighbor coupled lasers is generated by end-pumping the Nd:YAG laser crystal with three identical high power Ar^+ laser ($\lambda = 514.5$ nm) beams. In the observations reported here, the coupled lasers display chaotic dynamics; the coupling strength decreases for larger separation distances [13,15]. The intensities of the three coupled lasers are sampled individually with three photodetectors and a digital oscilloscope.

Experimental intensity measurements are displayed in Fig. 1 for a distance of 0.64 mm between adjacent pump beams. The two outer lasers in the array (lasers 1 and 3) have nearly identical intensity fluctuations and power spectra. Chaotic bursting is present in the intensity time series of the coupled elements [Figs. 1(a)–1(c)], which are thus of a distinctly nonstationary nature. The corresponding power spectra are shown in Figs. 1(d)–1(f). Plotting the intensity of laser 3 versus that of laser 1 [Fig. 1(h)], we find near-identical synchronization between the chaotic signals of the two outer lasers. However, the outer lasers are able to interact only with each other through the intermediary center laser (laser 2) since only nearest-neighbor coupling is present. Figures 1(g) and 1(i) are plots of the intensity time series for the center laser (laser 2) versus those of the outer lasers (lasers 1 and 3). No synchronization relationship is obvious, even though the center laser mediates the identical synchronization of the outer lasers. If one is given only the time series of one of the two outer lasers and the time series of the middle laser, how can we test for interdependence between these time series? This is the question we address in the remainder of

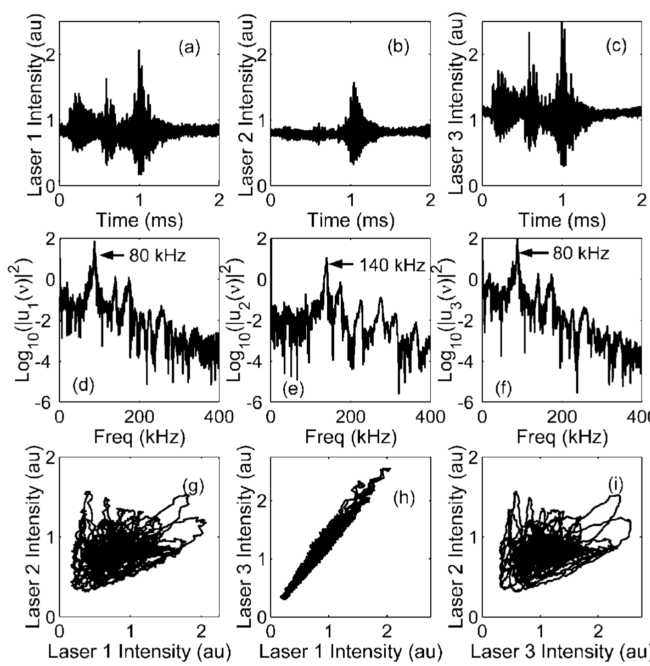

FIG. 1. (a)–(c): Experimental intensity time series (25 000 data points, sampled every 80 ns), showing chaotic bursts and similarity of the dynamics of lasers 1 and 3; (d)–(f): corresponding power spectra. Note the similarity of the spectra of the outer lasers and difference of these with respect to the central laser; (g)–(i): synchronization plots of the intensity time series for pairs of lasers. Lasers 1 and 3 are synchronized identically, while laser 2 is not obviously synchronized to the outer lasers.

this paper, and for this purpose we use the concept of phase synchronization.

As discussed above, an analytic phase, $\Phi_A(t)$, and a Gaussian filtered phase, $\Phi_G(t)$, are used. For an experimental data set, $V_{A,G}(t)$ is generated by applying an appropriate fast Fourier transform algorithm to $I(t)$, multiplying the result $u(\nu)$ by $f(\nu)$ and applying an inverse fast Fourier transform. $\Phi_{A,G}(t)$ is defined to be continuous in time, i.e., as $\Phi_{A,G}(t)$ increases through 2π it is not discontinuously set equal to zero. Thus $[\Phi_{A,G}(t) - \Phi_{A,G}(0)]/2\pi$ represents the number of counterclockwise rotations executed between time 0 and time t by the complex number $V_{A,G}(t) - \langle V_{A,G}\rangle$.

Phase synchronization between an outer laser and the center laser (lasers 1 and 2) is shown by plotting their relative phase versus time (Fig. 2). Figure 2(a) shows the difference of the analytic signal phases $\Delta\Phi_A(t)$ for these lasers, which has a large range of variation ($\sim$130 rotations). Phase synchronization is not discernible. Next, in Fig. 2(b), the difference of $\Phi_G(t)$ for these two lasers is plotted; these phases were calculated with $\nu_o = 140$ kHz (solid line) or $\nu_o = 80$ kHz (dotted line), and $\sigma = 15$ kHz. These ν_o were selected as the power spectra display significant peaks at these values. The product of σ and the length of the time series is much greater than unity, ensuring a physically significant phase for the given data. Synchronization of the side and central lasers in the frequency regime of $\nu_o = 140$ kHz is immediately apparent as the flat portion of this plot extends across essentially the entire time of observation (solid line). Periods of phase synchronization and phase slipping are found in the less correlated frequency regime of $\nu_o = 80$ kHz (dotted line). No indication of synchronization is found when one of the component phases of $\Delta\Phi_G(t)$ for $\nu_o = 140$ kHz, $\sigma = 15$ kHz, is replaced with a surrogate phase extracted from another experimental data set taken from this array under identical conditions [dashed line in Fig. 2(b)]. We note that plots similar to that in Fig. 2(a) showing no synchronization were also obtained for the other definitions of phase mentioned in [17].

The benefits of selecting dynamics of specific frequency ranges with a Gaussian filter are evident also from a normalized probability distribution of relative phases ($\Delta\Phi \bmod 2\pi$). Figure 3 displays a large peak for the Gaussian filtered phase ($\nu_o = 140$ kHz and $\sigma = 18$ kHz), while the analytic signal phase shows only a smaller, much broader peak. The surrogate cases display nearly uniform distributions.

A quantitative measure of synchronization is obtained by computing entropies from probability distributions of the phase differences (entropies were introduced in [6] to quantify phase synchronization). We are able to quantify the extent of phase synchronization for different frequency ratios by moving the center of the Gaussian filter over the frequency range of interest. Figure 4 displays this method for visualization of phase synchronization between lasers 1 and 2. Significant phase synchronization is seen only for

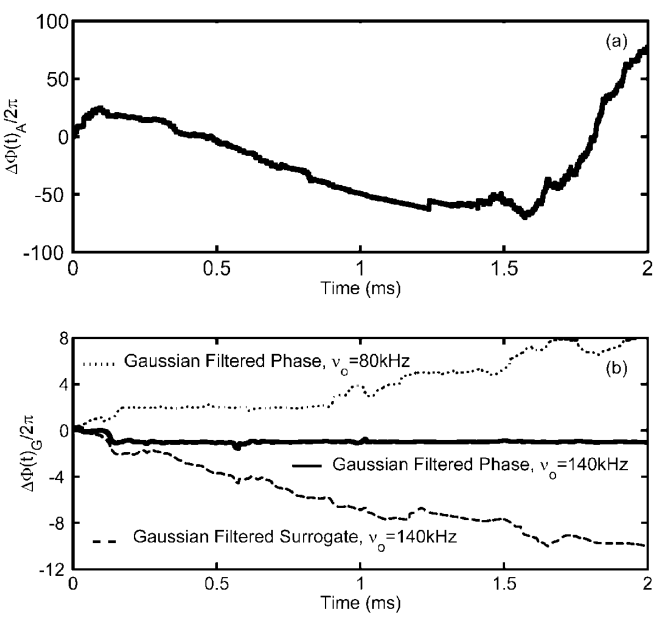

FIG. 2. Time series for the relative phases (a) $\Delta\Phi_A(t)$ and (b) $\Delta\Phi_G(t)$ for lasers 1 and 2. Synchronization is not discernible in (a), while (b) shows phase synchronization when the Gaussian filter is centered at $\nu_0 = 140$ kHz. Intermittent periods of phase synchronization are observed with $\nu_o = 80$ kHz (dotted line). The dashed line is the relative phase computed with a surrogate time series; as expected, it does not show synchronization. In all cases in (b) $\sigma = 15$ kHz for the Gaussian filter.

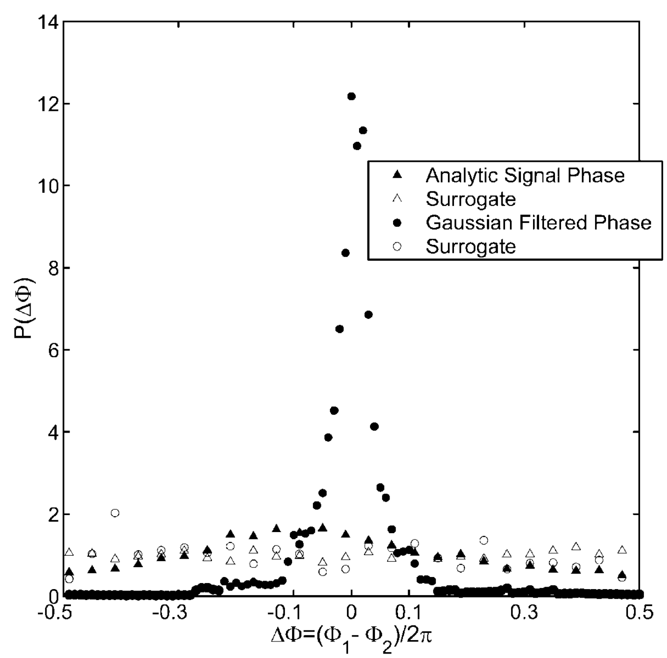

FIG. 3. Normalized probability distribution of relative phases. The Gaussian ($\nu_o = 140$ kHz, $\sigma = 18$ kHz) phase distribution shows a large peak compared with the analytic signal phase distribution. Computation with surrogate phases yields nearly uniform distributions (open symbols).

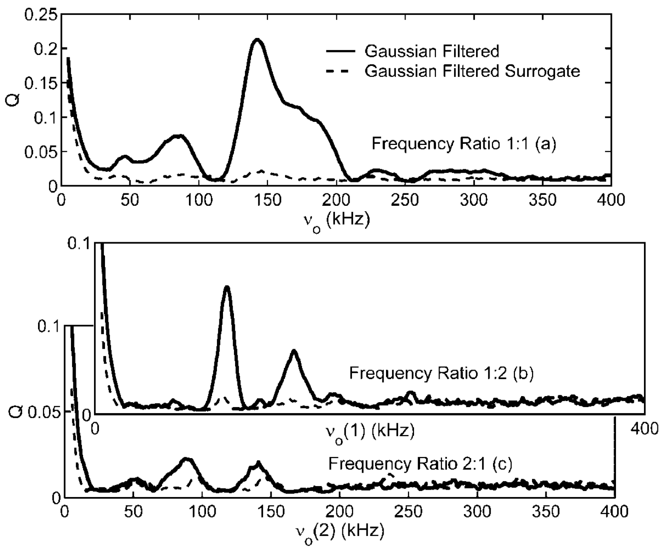

FIG. 4. Phase synchronization of laser 1 with laser 2 for frequency ratios (a) 1:1, (b) 1:2, and (c) 2:1. In all cases $\sigma =$ 18 kHz. Good 1:1 synchronization occurs in the range 125–200 kHz (a), while 1:2 and 2:1 synchronization are weaker and occur over more restricted frequency ranges (b) and (c). The dashed lines are the results of computations with surrogate time series.

frequency ratios of 1:1 [Fig. 4(a)], 1:2 [Fig. 4(b)], and 2:1 [Fig. 4(c)]. The synchronization measure $Q = 1 - S_{\mathrm{rel}}$ is used, where S_{rel} is the normalized entropy [6] given by $S_{\mathrm{rel}} = \sum_{j=1}^{n} P_j ln P_j / ln(1/n)$ (where P_j is the probability of $\Delta\Phi_G$ mod2π being in bin j). When a surrogate time series (dashed lines in the figures) is introduced for one of the laser phases, the phase synchronization peaks disappear.

In conclusion, our work illustrates that the detection of phase synchronization may require careful consideration of the nature of the time series measured. In the experimental example considered here, the time series are of a distinctly nonstationary nature, and it is clearly advantageous to introduce a Gaussian filtered phase variable. We are then able to quantitatively assess phase synchronization for different frequency components of the dynamics, which would not have been discernible otherwise. This technique should be widely applicable in situations of physical and biological interest.

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[17] Other choices that give previously mentioned means of defining a phase are $f_1(\nu) = 1 - ie^{i\nu T}$, yielding $V(t) = I(t) - iI(t - T)$, and $f_2(\nu) = 1 + \nu\alpha$, yielding $V(t) = I(t) + i\alpha \frac{dI(t)}{dt}$. Again, both of these choices may be thought of emphasizing positive frequencies: $|f_2(\nu)| > |f_2(-\nu)|$ for $\nu > 0$ (we assume $\alpha > 0$), and for a spectrum peaked at $\nu \sim \frac{\pi}{2T}$ we note that $f_1(\frac{\pi}{2T}) = 2$ while $f_1(-\frac{\pi}{2T}) = 0$.
[18] One could use other bandlimited forms for $f(\nu)$ and obtain results similar to our Gaussian choice. For example, we have found that this is the case for $f(\nu) = (2\sigma)^{-1}\theta(\sigma - |\nu - \nu_o|)$. In this case we calculated $V(t)$ by Fourier transforming $I(t)$, replacing $u(\nu)$ by $f(\nu)u(\nu)$, and inverse transforming. In the case that the calculation is done directly in time, via a convolution, the convolution is numerically easier to perform for the Gaussian choice since the Gaussian kernel is better localized in time.
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