Resonant Emittance Transfer Driven by Space Charge

I. Hofmann and O. Boine-Frankenheim
GSI Darmstadt, Planckstrasse 1, 64291 Darmstadt, Germany
(Received 2 March 2001; published 26 June 2001)

Space charge can lead to emittance and/or energy exchange known as "equipartitioning issue" in linacs, or space-charge coupling in high-current synchrotrons. It is described here as an internal resonance driven by the self-consistent space-charge potential of coherent eigenmodes. By a detailed comparison of analytical theory with 2D particle-in-cell simulation for Kapchinskij-Vladimirskij (KV) and waterbag distributions, we discuss characteristic features of this resonance mechanism in the vicinity of the symmetric focusing resonance band—for practical purposes, the most important case—and discuss the applicability of the linearized KV theory.

DOI: 10.1103/PhysRevLett.87.034802 PACS numbers: 41.75.–i, 29.27.Bd, 29.17.+w

The issue of energy/emittance exchange in space-charge dominated beams is of increasing interest in connection with new projects of high-power proton linear accelerators as well as high-current synchrotrons. In recent simulation studies of 3D bunched linac beams, it was shown that nonequipartitioned beams are not subject to energy/emittance exchange in certain regions of parameter space, while in other regions (associated with internal resonances) energy exchange in the direction of equipartition is inevitable [1]. The underlying mechanism—noting that Coulomb (intrabeam) scattering is too slow—is collective oscillations of the space-charge density that creates nonlinear forces similar to those of magnetic sextupoles or octupoles and leads to the possibility of resonant coupling. A self-consistent description of this mechanism of collectively driven resonances has first been proposed in Ref. [2] using a Vlasov perturbation approach to the anisotropic 2D Kapchinskij-Vladimirskij (KV) distribution. Exponential growth of properly defined eigenmodes of density oscillations provides the source of coupling. In realistic non-KV beams, the required nonlinearity may be already present in the initial distribution which enhances the speed of the exchange. The successful comparison of KV theory with the 3D simulation cases in Ref. [1] has in part motivated the present 2D simulation study, which aims at a detailed analysis of the range of validity of the linearized KV theory as well as the effect of a realistic distribution function. Furthermore, the present 2D study may be of direct relevance to explore coherent space-charge resonances and beam halos in synchrotrons and proton driver rings—a subject recently addressed by different authors [3–5]. We focus on an extended region around the case of symmetric focusing constants which appears to be the practically most significant resonance. This expands the well-known single particle "Montague resonance" [6] to the self-consistent coherent case and establishes its equivalence to the corresponding 3D linac resonance. Furthermore, our study clarifies issues on equipartition and thermodynamics raised in recent literature by Kishek *et al.* [7].

The basis of our comparison is the analytical calculation of growth rates for leading order eigenmodes from the dispersion relations in Ref. [2], which are based on perturbations of the anisotropic KV distribution in transverse phase space,

$$f_0(x, y, p_x, p_y) = \frac{NT\nu_y/\nu_x}{2\pi^2 m\gamma a^2} \times \delta\left(H_{0x} + TH_{0y} - m\gamma\nu_x^2\frac{a^2}{2}\right), \tag{1}$$

in a constant focusing system with arbitrary focusing ratio and emittance ratio. The (initially) decoupled Hamiltonians, H_{0x}, H_{0y}, are constants of the motion; a is the beam radius in x (similarly b in y) and ν_x, ν_y the corresponding betatron tunes. The energy anisotropy T is the ratio of oscillation energies in x and y which can be written for harmonic oscillators as

$$T \equiv \frac{a^2\nu_x^2}{b^2\nu_y^2} = \frac{\epsilon_x\nu_x}{\epsilon_y\nu_y}. \tag{2}$$

Solution of the linearized Vlasov equation leads to dispersion relations for eigenmodes with space-charge potentials expanded in polynomials in x and y. Eigenmodes are characterized by the order of the polynomial, and exponential instability is found in certain regions of parameter space. As the unperturbed KV beam has no coupling, some small initial density fluctuations lead to finite nonlinear coupling forces if the eigenmode grows exponentially. Note that the energy anisotropy leads to a substantial increase in the number of eigenmodes for given order compared with the isotropic KV theory [8] or—even more—with fluid models [9].

Solutions of the dispersion relations can be plotted in terms of the betatron tune ratio ν_x/ν_y (for unequal emittances different from the zero space-charge tune ratio ν_{0x}/ν_{0y}) which relates to the appearance of resonance, and the tune depression in one direction as a measure for space charge. Results are shown in Fig. 1 for an emittance ratio $\epsilon_x/\epsilon_y = 2$, and constant tune depression in the more space-charge dominated y direction, here $\nu_y/\nu_{0y} = 0.8$.

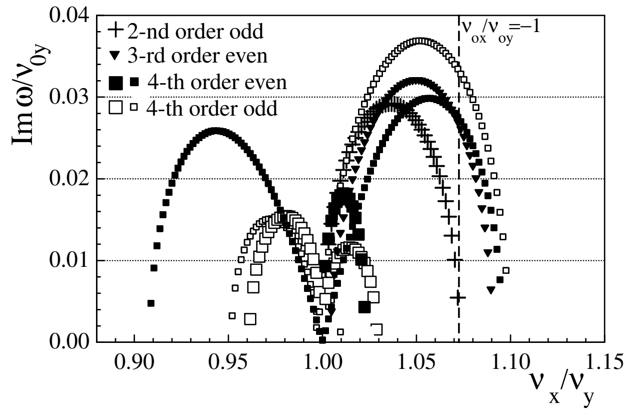

FIG. 1. Analytic instability growth rates (normalized) for constant $\nu_y/\nu_{0y} = 0.8$ and $\epsilon_x/\epsilon_y = 2$ as a function of the betatron tune ratio, and for different modes up to fourth order.

Small markers relate to oscillatory modes ($\mathrm{Re}\omega \neq 0$), big markers to nonoscillatory modes ($\mathrm{Re}\omega = 0$). Note that the width of the stop band $0.91 \lesssim \nu_x/\nu_y \lesssim 1.1$ is quite symmetric about the tune ratio unity, where all rates vanish; writing it in terms of the zero space-charge tune ratio, $0.88 \lesssim \nu_{0x}/\nu_{0y} \lesssim 1.02$, it is unsymmetric due to the space-charge tune shift. Growth rates are in units of ν_{0y}; hence, the typical growth rate of 0.03 in Fig. 1 corresponds to $(2\pi 0.03)^{-1} \approx 5.3$ zero space-charge betatron wavelengths in the y direction. The dominant modes appear to be of fourth order, which would suggest a resonance condition $2\nu_x - 2\nu_y \approx 0$ in the limit of small space charge. This resonance was the subject of the single-particle approach by Montague from which it was concluded that equal tunes need to be avoided in a synchrotron [6]. Our self-consistent approach includes the time evolution of the corresponding mode; furthermore it shows, perhaps unexpected, the appearance of unstable third order modes for $\nu_x/\nu_y \gtrsim 1$, which in a single-particle model would appear only near $2\nu_x - \nu_y \approx 0$, or $\nu_x - 2\nu_y \approx 0$. In addition, Fig. 1 indicates also instability of a second order "skew" or "tilting" mode driven by a space-charge xy term if the conditions $\nu_x/\nu_y > 1$ and $\nu_{0x}/\nu_{0y} < 1$ are satisfied (for details see Ref. [2]).

To explore the significance of these resonances, we have employed a standard 2D particle-in-cell code with 10^5 simulation particles in a rectangular conducting boundary sufficiently far away from the beam. It is essential for KV simulations with many particles to initialize the simulations with finite (but still small) initial density fluctuations for all modes under consideration. A quiet start technique, likewise a very large number of randomly chosen particles, causes artificial delay of the onset of modes which can be avoided by imprinting on the uniform initial density a random fluctuation spectrum at the percent level. We have carried out simulations with initial KV as well as rms (but not intrinsically) matched waterbag distributions (WB) for the same parameter range as in Fig. 1. In Fig. 2 we show typical examples of dynamical evolution of rms emittances at tune ratios—slightly different for KV and WB—where the emittance exchange is maximum. It is noted that for the KV beam this occurs for symmetric focusing with full equipartitioning (equal emittances in Fig. 2); the WB shows visible overshoot beyond equipartition for a slightly smaller tune ratio and leads to a final emittance ratio 0.8. It should be noted that the stability of the saturated emittances is consistent with theory since for $\epsilon_x/\epsilon_y = 0.8$ the growth rate graphs are inverted about $\nu_x/\nu_y = 1$ and reduced in width; hence, no unstable modes are covered. Also, this overshoot is restricted to a small interval of tune ratios at 1.05 ± 0.01. The simulation growth rates found compare well with the analytical predictions of maximum e-folding times of about five betatron periods.

Results for the saturated rms emittance growth in each plane are given in Fig. 3. For the KV case, the full stop band width is in excellent agreement with the analytical one. The predicted absence of emittance exchange for $\nu_x/\nu_y = 1$ is also confirmed by the simulation. Left from this point, the emittance transfer is, perhaps surprisingly, into the direction of the originally larger emittance (x). The maximum exchange is reached at $\nu_{0x}/\nu_{0y} = 1(\nu_x/\nu_y \approx 1.073)$, where the tilting mode ceases to be unstable by theory. For the WB, emittance transfer is limited to an even narrower stop band, though with different features. The left edge of the stop band coincides with the disappearance of nonoscillatory modes for $\nu_x/\nu_y \lesssim 0.96$; likewise, only negligible emittance exchange is found for $\nu_x/\nu_y \gtrsim 1.073$. At this symmetric focusing, and for slightly smaller tune ratio, the WB is characterized by the appearance of the tilting mode: emittances are periodically exchanged between x and y, similar to a second order difference resonance driven by skew quadrupoles. There is only a small irreversible approach to equipartition which is indicated by the markers. These simulations suggest that

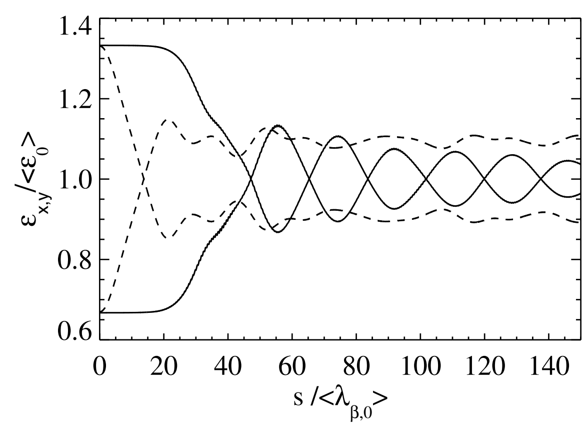

FIG. 2. Evolution of rms emittance growth factors for initial KV (solid line) with $\nu_x/\nu_y \approx 1.073$, and WB (dotted line) with $\nu_x/\nu_y \approx 1.06$. Units on the abscissa are betatron periods in y in the absence of space charge.

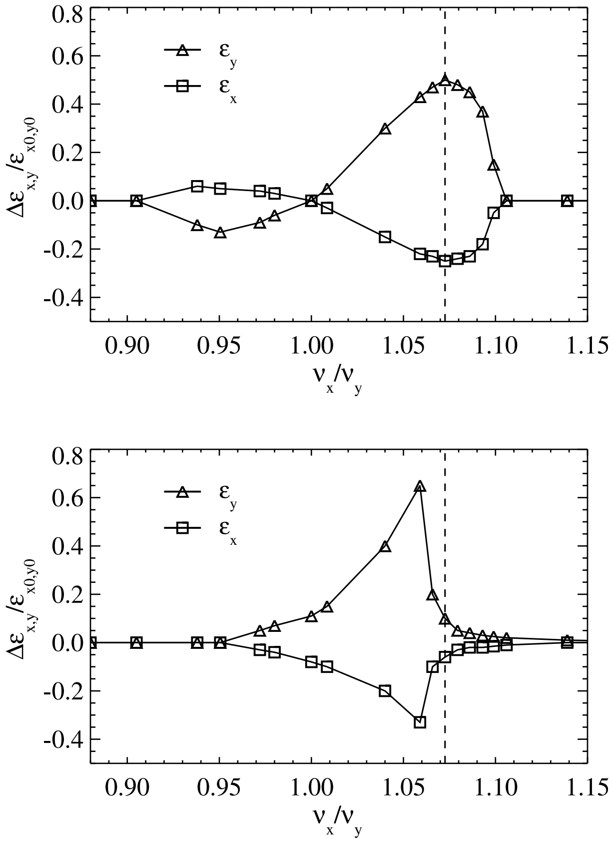

FIG. 3. Simulation results for initial KV (top) and WB (bottom) distributions ($\nu_y/\nu_{0y} = 0.8$, $\epsilon_x/\epsilon_y = 2$) as a function of the tune ratio (dashed vertical line: symmetric focusing).

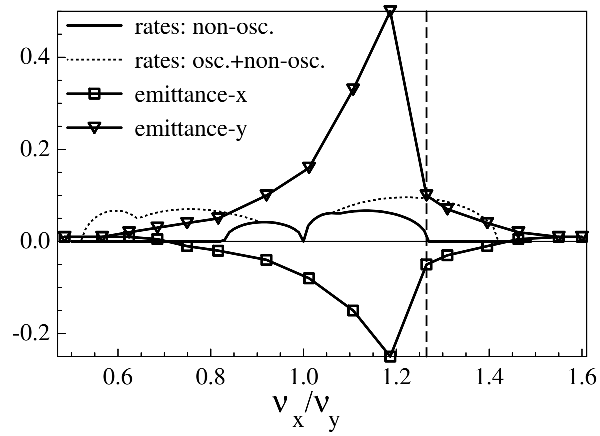

FIG. 4. Emittance growth factors (units $\Delta\epsilon/\epsilon$) for WB simulation compared with maximum analytical growth rates (units as in Fig. 1) for $\nu_y/\nu_{0y} = 0.5$, $\epsilon_x/\epsilon_y = 2$; dashed vertical line is symmetric focusing.

for realistic beam models (such as WB) only nonoscillatory modes contribute to emittance exchange, whereas oscillatory modes appear to be ignorable. A possible explanation might be that in non-KV beams a finite spread of single-particle frequencies leads to Landau damping and suppression of instability only for modes oscillating with a frequency near the proper harmonic of single-particle frequencies, whereas nonoscillatory modes remain unaffected.

We have compared this with a semi-Gaussian (SG) distribution—uniform in real space, Gaussian in velocity space—and found a behavior nearly symmetric about the tune ratio 1.05—very close to that of the WB below 1.05.

Analytical growth rates for stronger tune depression show a quite similar distribution of modes as in Fig. 1 as long as $\nu_y/\nu_{0y} \gtrsim 0.5$, besides overall increasing values for the rates and broadening of the stop band. In Fig. 4 ($\nu_y/\nu_{0y} = 0.5$) we compare WB simulation growth factors with analytical rates by plotting the maxima of growth rates (without detailed mode distinction) including all modes as well as those from only nonoscillatory modes. These simulation results confirm the observation made above that non-negligible emittance transfer is limited to the nonoscillatory modes: for $\nu_x/\nu_y \approx 0.75$ in Fig. 4, where the oscillatory fourth order mode adopts its maximum growth rate, we have found only 4% emittance growth in y, and 1% decrease in x. As this occurs right at the beginning of the simulation, we attribute this small effect to the conversion of nonlinear field energy for which theory predicts, at this tune depression, 2% emittance growth in both planes of an rms matched WB beam of circular cross section [10]. For the KV simulation of the same case, the emittance change is somewhat larger ($\approx$3 times), but in the opposite direction. It is close to that found in Fig. 3 for the corresponding point, $\nu_x/\nu_y \approx 0.94$, where the same oscillatory fourth order mode adopts its maximum growth rate.

Similarly, the large growth rates of oscillatory modes at the right end of the stop band (beyond symmetric focusing) do not lead to noticeable emittance transfer in the simulation. We use these conclusions to define "effective stop bands" from the analytical dispersion relations by ignoring contributions from unstable oscillatory eigenmodes. This expands the stable regions visibly for a small emittance ratio; for $\epsilon_x/\epsilon_y = 5$ or larger, the unstable regions of oscillatory modes are already contained inside those of nonoscillatory modes and the distinction "effective stop bands" becomes unnecessary. A further important finding is the absence of additional stop bands for higher than fourth order modes in the simulation. In particular, Fig. 4 shows that there is no evidence of emittance transfer for the WB simulations near tune ratios $\nu_x/\nu_y \approx 2/3, 3/2$, where fifth order stop bands could be expected (the KV simulation, instead, has indicated weak resonances).

In Fig. 5 we show the overall picture of these effective stop bands for different emittance ratios. Growth rates pertain to the maxima of all nonoscillatory modes of either second (tilting), third, or fourth order. Maxima become as large as one (zero space charge) betatron period for large tune depression and emittance ratio (bottom of Fig. 5). Note that the parameter range of circular accelerators is typically limited to $\nu_y/\nu_{0y} > 0.9$ (stop bands in Fig. 5 actually extend to the zero space-charge limit at

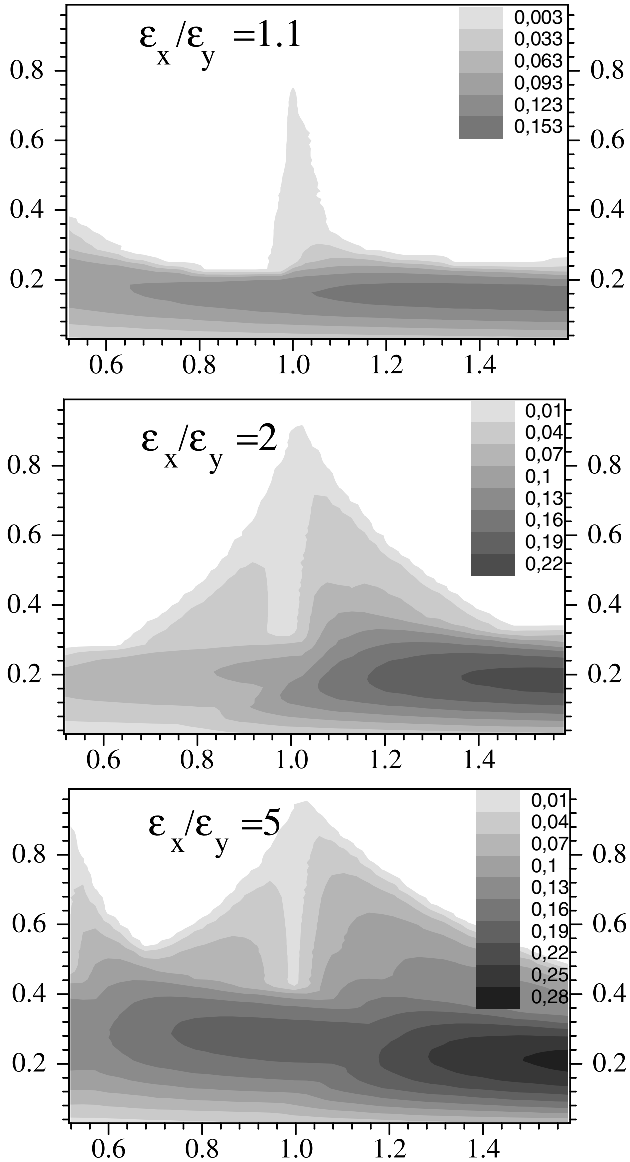

FIG. 5. Stability charts with analytically calculated effective stop bands (only nonoscillatory modes) for arbitrary ν_y/ν_{0y} (ordinate), tune ratios ν_x/ν_y (abscissa), and different emittance ratios. Growth rates (grey scales in equidistant steps) in units of zero space-charge betatron tune (in y).

$\nu_x/\nu_y = 1$, with growth rates and widths decreasing to zero, which is not resolved by the discrete steps of the contour plots); in high-current linacs the range of interest is typically $0.5 < \nu_y/\nu_{0y} < 0.8$, where y should stand for the direction in which the linac bunch emittance is smaller.

The effective stop band at tune ratio $\nu_x/\nu_y \approx 1$ vanishes completely for $\epsilon_x/\epsilon_y = 1$ due to vanishing energy anisotropy. Likewise, the stop band at tune ratio 1/2 disappears for $\epsilon_x/\epsilon_y = 2$. The resonance structure gets lost below a sufficiently strong tune depression, depending on the emittance ratio, due to a complete overlap with neighboring resonance bands. For the large emittance ratio of five the resulting “sea of instability” is raised to $\nu_y/\nu_{0y} \lesssim 0.5$, with a strong contribution to the growth rate from the overlapping band of third order modes originating at $\nu_x/\nu_y \approx 0.5$. In this region approach to equipartition is predicted for all tune ratios. It is noted that growth rates drop to zero when approaching the space-charge limit. For $\nu_y/\nu_{0y} = 0$, betatron tunes in both planes vanish; hence, any coupling occurs at zero rate.

The finding of energy exchange confined to resonance stop bands for both the KV and WB (also SG) beams clarifies that equipartition is not the natural state towards which real beams evolve as suggested in Ref. [7] in the context of symmetric focusing. We find, instead, that for moderate tune depression and not excessive emittance ratios—pertinent to most applications—the area of parameter space where anisotropic beams resist equipartition is actually the dominant one. Our 2D studies give strong support for the thesis that “islands of stability” for realistic beams are well described by regions where growth rates from nonoscillatory KV modes of second, third, or fourth order are absent. The relatively weak influence of the shape of the distribution function on the extent of the stable regions may be understood by appreciating that the primary source of free energy driving the resonances is the amount of anisotropy between different degrees of freedom, and not the distribution of energies within a degree of freedom. These conclusions reaffirm the proposal by Jameson [11] to use these stability regions for the design of nonequipartitioned high intensity linacs.

[1] I. Hofmann, J. Qiang, and R. Ryne, Phys. Rev. Lett. **86**, 2313 (2001).

[2] I. Hofmann, Phys. Rev. E **57**, 4713 (1998).

[3] M. Venturini and R. L. Gluckstern, Phys. Rev. ST Accel. Beams **3**, 034203 (2000).

[4] J. A. Holmes, V. V. Danilov, J. D. Galambos, D. Jeon, and D. K. Olsen, Phys. Rev. ST Accel. Beams **2**, 114 202 (1999).

[5] A. V. Fedotov *et al.,* in *Proceedings of the European Particle Accelerator Conference, Vienna, 2000,* p. 1289.

[6] B. W. Montague, CERN-Report No. 68-38, CERN, 1968.

[7] R. A. Kishek, P. G. O’Shea, and M. Reiser, Phys. Rev. Lett. **85**, 4514 (2000).

[8] R. L. Gluckstern, in *Proceedings of the Linac Conference, 1970* (Fermilab, Batavia, IL, 1970), p. 811.

[9] S. M. Lund and R. C. Davidson, Phys. Plasmas **5**, 3028 (1998).

[10] T. P. Wangler, K. R. Crandall, R. S. Mills, and M. Reiser, IEEE Trans. Nucl. Sci. **32**, 2196 (1985).

[11] R. A. Jameson, in *Proceedings of the Third Workshop on Advanced Accelerator Concepts, Port Jefferson, 1992,* AIP Conf. Proc. No. 279 (AIP, New York, 1993), p. 969.