Number of Fermion Generations Derived from Anomaly Cancellation

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We prove that global anomaly cancellation requires more than one generation of quarks and leptons, provided that the standard model fields propagate in two universal extra dimensions. Furthermore, if the fermions of different generations have the same gauge charges and chiralities, then global anomaly cancellation implies that there must be three generations.

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The existence of three generations of quarks and leptons is a major source of bafflement for particle physics. By contrast, the particle content within a generation is constrained by the mathematical structure of the standard model. Local anomaly cancellation substantially reduces the arbitrariness in choosing the $SU(3)_C \times SU(2)_W \times U(1)_Y$ charges of the fermions [1]. For example, the existence of the observed quarks requires the leptons to cancel the $SU(2)_W \times U(1)_Y$ triangle anomalies. Furthermore, the $SU(2)_W$-doublet lepton cancels the global anomaly [2] of the $SU(2)_W$-doublet quark within each generation.

In this Letter, we show that the number of generations may also be determined by the anomaly cancellation conditions. In order for this to happen, we are led to consider the existence of extra spatial dimensions accessible to all the standard model particles. If the number δ of these "universal" extra dimensions is odd, then there are no local gauge anomalies in the $(4 + \delta)$-dimensional theory. [For odd δ, the higher-dimensional analog of the three-dimensional Redlich anomaly [3] could spoil gauge invariance of the quantum effective action. However, the anomalous variation of the action can be canceled by a Chern-Simons term (see [4] for a discussion of the $\delta = 1$ case).] Therefore, additional anomaly cancellation conditions that may restrict the number of generations could arise only for even δ. The natural choice is then $\delta = 2$. Current experimental data impose a rather loose upper bound $R \gtrsim (0.5\ \text{TeV})^{-1}$ on the size of two universal extra dimensions [5].

The Lorentz group in six dimensions has spinorial representations of definite chirality with four components. A representation of the 8×8 anticommuting gamma matrices, Γ^α with $\alpha = 0, \ldots, 5$, is given in Ref. [6]. The Γ_7 matrix, analog to the γ_5 matrix in four dimensions, has eigenvalues ± 1 corresponding to the six-dimensional fermion chiralities. A six-dimensional chiral fermion, upon compactification on a smooth manifold (without magnetic fluxes [7]) to four dimensions, gives rise to vectorlike fermions. A four-dimensional theory with chiral fermions can be obtained by compactifying the two extra dimensions on an orbifold, for example, the T^2/Z_2 orbifold constructed in Ref. [5]. This orbifold gives rise to a chiral four-dimensional theory by projecting out half of the components of the six-dimensional Weyl fermions, while the gauge group in four dimensions is the same as in six dimensions—the standard model gauge group.

We assume that the six-dimensional theory is chiral and free of irreducible local as well as global gauge anomalies. Furthermore, the reducible anomalies are canceled by the Green-Schwarz mechanism [8], which is a generic feature of six-dimensional theories [9]. Our main results refer to the nonsupersymmetric standard model in six dimensions, but we also discuss supersymmetry towards the end, where we point out that it is hard to cancel the anomalies in this case.

We emphasize that we consider the six-dimensional theory and the orbifold construction in an effective low-energy field theory framework. It would be very interesting to find an explicit string theory realization of this nonsupersymmetric field theory. In this context, we note the assumption that there are no "twisted sector" chiral four-dimensional fermions localized at the fixed points and charged under the gauge group. These commonly arise in heterotic orbifolds [10], but not in open-string orbifolds [11]; this suggests that the place to look for string realization might be a type-I construction, or, perhaps even a more exotic construction involving little string theory. However, this lies beyond the scope of this note—all we aim here is to provide a consistent low-energy framework.

To this end, consider a generation of six-dimensional fermions, $\mathcal{Q}, \mathcal{U}, \mathcal{D}, \mathcal{L}, \mathcal{E}$, whose zero modes form a generation of four-dimensional quarks, $\mathcal{Q}^{(0)} \equiv (u, d)_L$, $\mathcal{U}^{(0)} = u_R, \mathcal{D}^{(0)} = d_R$, and leptons, $\mathcal{L}^{(0)} = (e, \nu_e)_L$, $\mathcal{E}^{(0)} = e_R$. The four-dimensional anomalies cancel automatically within a generation, and from a four-dimensional point of view this is sufficient for consistency. In what follows, we will assume that the four-dimensional theory is obtained as a deformation of a consistent (i.e., anomaly free) six-dimensional theory. We will show that the six-dimensional anomalies do not cancel so easily, and not only restrict the six-dimensional chiralities within a generation but also impose a constraint on the number of generations.

The local gauge anomaly in six dimensions is given by a square one-loop diagram (for a self-contained introduction to anomalies in six dimensions, see [12]). Consider first the anomalies of the unbroken $SU(3)_C \times U(1)_Q$ part of the gauge group. A necessary condition for the consistency of the six-dimensional theory is the cancellation of the irreducible gauge anomalies (i.e., which cannot be canceled by the Green-Schwarz mechanism [8] or its generalization [9] with multiple antisymmetric tensors), required for allowing the massless gluon and photon. The $U(1)_Q[SU(3)_C]^3$ gauge anomaly is the only irreducible one (the quartic anomaly is factorizable for SU(3) and SU(2), and irreducible for $SU(n)$ with $n \geq 4$), and imposing its cancellation within a generation we find that $\mathcal{Q}$ should have opposite chirality compared with $\mathcal{U}$ and $\mathcal{D}$.

The six-dimensional gravitational and mixed gauge-gravitational anomalies must also cancel to allow a massless graviton. The cancellation within one generation of the six-dimensional $U(1)_Q$-gravitational anomaly implies that $\mathcal{L}$ and $\mathcal{E}$ also have opposite chirality. The pure gravitational anomaly cancels only if the number of fermions with $+$ and $-$ chiralities is the same (in six dimensions, a self-dual antisymmetric tensor has gravitational anomaly equal to that of 28 Weyl fermions; hence, it cannot be used to cancel the gravitational anomaly of $\mathcal{L}$ and $\mathcal{E}$). As a result, there must exist an additional fermion, $\mathcal{N}$, with the same chirality as $\mathcal{E}$. The unprojected zero mode of $\mathcal{N}$ may be identified with a right-handed neutrino. The above arguments yield four possible chirality assignments of the fermions:

$$\mathcal{Q}_+, \mathcal{U}_-, \mathcal{D}_-, \mathcal{L}_-, \mathcal{E}_+, \mathcal{N}_+ \,, \tag{1}$$

$$\mathcal{Q}_+, \mathcal{U}_-, \mathcal{D}_-, \mathcal{L}_+, \mathcal{E}_-, \mathcal{N}_- \,, \tag{2}$$

and the ones obtained by interchanging $+$ and $-$. With these assignments, the reducible anomalies involving $U(1)_Q$ and $SU(3)_C$ also vanish, because the fermion representations are vectorlike under these groups.

The $SU(2)_W \times U(1)_Y$ six-dimensional anomalies do not cancel with the standard model field content, but this may not be troublesome because the electroweak symmetry is broken. In other words, one could speculate that the $SU(2)_W \times U(1)_Y$ anomalies from the underlying higher-dimensional theory would be responsible for part (or even all) of the W and Z masses. Nevertheless, embedding the six-dimensional theory in a consistent high-energy theory that includes quantum gravity most likely requires the $SU(2)_W \times U(1)_Y$ anomalies to be canceled within that underlying theory. This can be achieved through the Green-Schwarz mechanism: the $[SU(2)_W]^4$, $[U(1)_Y]^4$, $[SU(2)_W]^2[SU(3)_C]^2$, $[SU(3)_C]^2[U(1)_Y]^2$, and $[SU(2)_W]^2[U(1)_Y]^2$ anomalies are canceled by two antisymmetric tensor fields with appropriate Green-Schwarz couplings. [The cubic anomaly for SU(2) is identically zero, while the irreducible $[SU(3)_C]^3U(1)_Y$ anomaly vanishes within each generation.] We note that the presence of reducible anomalies is rather generic in six-dimensional chiral theories that embed the standard model; hence, antisymmetric tensors are a likely ingredient of any realistic six-dimensional model. (Some components of the antisymmetric tensors survive the orbifold projection and have axionlike couplings in the four-dimensional theory. They may acquire masses after the compactification, e.g., from terms localized at the orbifold fixed points, or they could provide a solution to the strong CP problem.)

The main point we make here is that there is an additional constraint. In six dimensions there are global gauge anomalies, analogous to the four-dimensional Witten anomaly [2]. They occur only for SU(3) [13], as well as SU(2) and G_2 gauge theories [14]; see also [15]. Global anomalies are due to the change of sign of the Weyl fermion determinant under gauge transformations that are topologically disconnected from the identity; in six dimensions these arise whenever the gauge group G has nontrivial $\pi_6(G)$ (the homotopy group of maps of the six-sphere onto the gauge group). The mathematical consistency of the theory requires these to cancel. Since the six-dimensional $SU(3)_C$ fermion representations are vectorlike, the $SU(3)_C$ global anomaly is canceled within each generation. On the other hand, the $SU(2)_W$ global anomaly cancellation condition [15] requires

$$N(2_+) - N(2_-) = 0 \bmod 6 \,, \tag{3}$$

where $N(2_\pm)$ is the number of doublets of chirality $\pm$. Since $N(\mathcal{Q}) = 3$ and $N(\mathcal{L}) = 1$, the $SU(2)_W$ global anomaly does *not* cancel within one generation for any chirality assignment. We are led then to consider the case of n_g generations with identical chirality assignments. The assignments obtained above, (1) and (2), give

$$n_g = 0 \bmod 3 \,. \tag{4}$$

This is a remarkable result. It is a compelling theoretical explanation for the existence of three generations. Although anomaly cancellation in six dimensions allows the number of generations to be a multiple of three, the only reasonable prediction is $n_g = 3$: a world with $n_g = 0$ would be rather dull, while $n_g \geq 6$ would imply that the gauge couplings blow up very fast above the compactification scale.

For $n_g = 3$, the effective six-dimensional theory is perturbative and well defined for a range of energies above $1/R$. The Kaluza-Klein modes of the standard model in $\delta = 2$ universal extra dimensions contribute at each mass level with $2 \times (81/10, 11/6, -2)$ to the one-loop coefficients of the β functions for the $U(1)_Y$, $SU(2)_W$, and $SU(3)_C$ gauge couplings. It follows that the six-dimensional standard model gauge interactions become nonperturbative at a scale $\sim 5/R$ [5]. The heavy states of string theory may become relevant at that scale if the other four extra dimensions have a large volume [16]. Alternatively, it is conceivable that the six-dimensional $SU(3)_C \times SU(2)_W \times U(1)_Y$ gauge couplings approach a strongly interacting fixed point in the ultraviolet [17], so that the scale of quantum gravity need not be lowered much below the Planck scale by large extra dimensions.

There is also some experimental evidence in favor of our $n_g = 3$ prediction: the existence of a fourth generation of chiral fermions is ruled out at the 97% confidence level (assuming no other physics beyond the standard model) by the electroweak precision measurements at LEP, SLD, and Tevatron [18]. Moreover, the number of light neutrinos that couple to the Z was measured at LEP to be very close to three. However, loopholes in these experimental constraints are not hard to imagine. For instance, the isospin-violating effects due to the Kaluza-Klein modes of the top quark give a positive T parameter [5], which in turn may allow a large mass splitting within the $SU(2)_W$-doublet fermions of a fourth generation. In this case, a chiral fourth generation would render an acceptable fit to the electroweak data. Likewise, the constraint on the number of $SU(2)_W$-charged neutrinos does not apply when they are heavier than half the Z mass. Hence, the determination of n_g from anomaly cancellation can be viewed as a prediction that will be tested in future experiments.

The anomaly cancellation conditions do not restrict the number of vectorlike generations. Even if these exist, there is a simple reason why they have not been seen yet: their masses are gauge invariant and are likely to be of the order of the fundamental (string) scale, $M_s > 1/R$. However, it is also possible to have a vectorlike six-dimensional generation and four-dimensional chirality introduced by the orbifold compactification such that the zero modes of the six-dimensional fermions form two chiral generations. Other ways of canceling the anomalies can also be found when the chirality assignments differ between generations. An example is two generations where one has chirality assignments given by Eq. (1), while the other's chirality is similar to that in Eq. (2). Depending on the four-dimensional chirality (L or R) assigned to the zero modes by the orbifold, there are two cases (up to the overall interchanges $+ \leftrightarrow -$ or $L \leftrightarrow R$):

$$\begin{aligned}&(\mathcal{Q}_+^1)_L, (\mathcal{U}_-^1)_R, (\mathcal{D}_-^1)_R, (\mathcal{L}_-^1)_L, (\mathcal{E}_+^1)_R, (\mathcal{N}_+^1)_R\,,\\ &(\mathcal{Q}_+^2)_R, (\mathcal{U}_-^2)_L, (\mathcal{D}_-^2)_L, (\mathcal{L}_+^2)_R, (\mathcal{E}_-^2)_L, (\mathcal{N}_-^2)_L\,,\end{aligned} \tag{5}$$

$$\begin{aligned}&(\mathcal{Q}_+^1)_L, (\mathcal{U}_-^1)_R, (\mathcal{D}_-^1)_R, (\mathcal{L}_-^1)_L, (\mathcal{E}_+^1)_R, (\mathcal{N}_+^1)_R\,,\\ &(\mathcal{Q}_+^2)_L, (\mathcal{U}_-^2)_R, (\mathcal{D}_-^2)_R, (\mathcal{L}_+^2)_L, (\mathcal{E}_-^2)_R, (\mathcal{N}_-^2)_R\,,\end{aligned} \tag{6}$$

where the upper index labels the generation. Case (5) gives rise only to vectorlike quarks and leptons in the effective four-dimensional theory. In case (6), however, the zero modes form two identical generations of chiral fermions. Thus, a more precise formulation of our result is that six-dimensional anomaly cancellation requires the existence of more than one fermion generation, and in the case of *identical* generations (i.e., same charges and chiralities, and also same properties under the orbifold transformation) their number has to be a multiple of three.

The results obtained thus far apply only to nonsupersymmetric theories. In the case of minimal supersymmetry in six dimensions, the anomalies are significantly more restrictive [9]. This is because $(1,0)$ supersymmetry requires all matter fermions to have the same chirality, opposite to that of the gauginos and gravitino. Canceling the anomalies by the Green-Schwarz mechanism severely constrains the matter content. Thus, for an SU(3) gauge theory with hypermultiplets only in the 3 and $\overline{3}$ representations, the cancellation of local and global anomalies combined requires that the number of hypermultiplets be 0, 6, 12, or 18; for SU(2) only 4, 10, or 16 doublets are allowed. [The upper limit holds in the theory without gravity. Using the antisymmetric tensor from the graviton supermultiplet to cancel the anomaly relaxes the upper limit and allows for a larger number of hypermultiplets, with the same periodicity, e.g., 24, 30, etc., for SU(3).] Since the number of fermions in the fundamental representation is $4n_g$ and $4n_g + 2$, for $SU(3)_C$ and $SU(2)_W$, respectively, this rules out the six-dimensional $(1,0)$ supersymmetric "standard model" with any n_g. Therefore, the supersymmetric models, often considered in the literature [19], with quarks and leptons in the bulk of two extra dimensions are anomalous. One could try the $n_g = 3$ case with two additional $SU(2)_W$-doublet hypermultiplets. This theory, however, suffers an irreducible $U(1)_Y[SU(3)_C]^3$ anomaly. This and the $U(1)_Y$-gravitational anomaly can be canceled simultaneously only if hypermultiplets with exotic $U(1)_Y \times SU(3)_C$ charges are added to the theory, which is a significant departure from the standard model. The higher supersymmetries in six dimensions (which reduce to $N = 4$ supersymmetry in four dimensions) do not allow (at least for now) a prediction regarding the number of generations: the $(1,1)$ supersymmetric theory is vectorlike, while the chiral $(2,0)$ theory remains rather mysterious. Hence, the compelling explanation for the existence of three fermion generations suggests that supersymmetry is broken at the string scale (or at least above the compactification scales of additional, smaller universal extra dimensions).

Another issue is whether the number of generations could be determined based on global anomaly cancellation conditions when the number of extra dimensions is larger. From the point of view of string theory only the cases $\delta = 2, 4, 6$ are interesting. Given that the $SU(3)_C$ representations are vectorlike within a generation, only $SU(2)_W$ could have a global anomaly. The relevant homotopy groups are $\pi_8[SU(2)] = Z_2$ for $\delta = 4$, and $\pi_{10}[SU(2)] = Z_{15}$ for $\delta = 6$ (see Ref. [20]). The generalization to $2 \le \delta \le 12$ of the global anomaly cancellation condition given in Eq. (3) is

$$c_\delta[N(2_+) - N(2_-)] = 0 \bmod n_\delta\,, \tag{7}$$

where c_δ is an integer, and n_δ is the number of homotopy group elements ($n_\delta = 12, 2, 15$ for $\delta = 2, 4, 6$.) Given that $N(2_+) - N(2_-)$ is even within each generation, there is no constraint on n_g when $\delta = 4$, while the global anomaly poses a severe restriction on n_g when $\delta = 6$. Only the case $\delta = 2$ is both predictive and viable, as a consequence of the fact that the homotopy group $\pi_6[\mathrm{SU}(2)] = Z_{12}$ is large and has an even number of elements.

An important point we should stress is that the global anomaly cancellation condition we have found applies also if the six-dimensional gauge group is larger, for example, if $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_W \times \mathrm{U}(1)_Y$ is embedded in a gauge group broken by the compactification. In such a scenario the global anomaly of $\mathrm{SU}(2)_W$ should appear as a local anomaly.

The arguments we have given apply for any size R of the two extra dimensions as long as there is a range of scales where an effective six-dimensional field theory is valid. However, the usual hierarchy problem suggests that the compactification scale should be close to the electroweak scale. One may view the derivation of the number of fermion generations based on anomaly cancellation conditions as evidence for the existence of two universal extra dimensions. Independent support for this conclusion is given by the successful breaking of the electroweak symmetry [6] by a composite Higgs field that arises due to standard model gauge dynamics in two universal extra dimensions.

In summary, we have shown that global anomaly cancellation for the standard model in two universal extra dimensions implies that there must be more than one generation of quarks and leptons, and, if these generations are identical from the point of view of the fermion charges, six-dimensional chiralities, and transformation properties under the orbifold projection, then their number should be three.

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