Level Quantization for the Noncommutative Chern-Simons Theory

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We show that the coefficient of the three-dimensional Chern-Simons action on the noncommutative plane must be quantized. Similar considerations apply in other dimensions as well.

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Chern-Simons (CS) field theories have been extensively investigated in various contexts since their appearance in physics literature as topological mass terms for odd dimensional gauge theories [1]. Chern-Simons theories on noncommutative spaces were introduced recently using the star product [2] or the operator formulation [3] and there have been a number of papers investigating the properties of such theories [4]. In the commuting case, it is well known that invariance of the theory under gauge transformations which are homotopically nontrivial requires the quantization of the coefficient of the CS term in the action, the so-called level number. An immediate and natural question is whether such a quantization would hold on a noncommuting space as well; this is the subject of this paper. This question was recently addressed in Ref. [5], where it was argued that there is no quantization of the level number for the noncommutative (NC) plane. We show that there is actually quantization of the level number on the plane. In fact, the result is stronger in the noncommutative case: there is quantization even for the U(1) theory. Consistency with the commutative limit is obtained in the following way. As the noncommutativity parameter θ approaches zero, the relevant transformations go over to the smooth homotopically nontrivial transformations in SU(N). However, if we interpret them as U(1) transformations, the limit is singular. There is then no need to demand invariance under these for the U(1) theory in the commutative limit, removing the reason for level quantization in this limit.

We follow the notation of Ref. [3]. The CS action in three dimensions is given in terms of covariant derivative operators D_μ by

$$S = \lambda 2\pi\theta \int dt\, \mathrm{Tr}\Big(i\,\frac{2}{3} D_\mu D_\nu D_\alpha + \omega_{\mu\nu} D_\alpha\Big)\epsilon^{\mu\nu\alpha}. \tag{1}$$

Here space consists of a noncommutative plane x^1, x^2 and a commutative third dimension $x^0 = t$, satisfying

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \tag{2}$$

with the antisymmetric $\theta^{\mu\nu}$ and $\omega_{\mu\nu}$ defined in terms of a c-number parameter θ as

$$\begin{aligned} \theta^{12} = -\theta^{21} = \theta, \qquad \theta^{01} = \theta^{02} = 0\,, \\ \omega_{12} = -\omega_{21} = -\frac{1}{\theta}\,, \qquad \omega_{01} = \omega_{02} = 0\,. \end{aligned} \tag{3}$$

A standard realization of the NC plane is given by the Fock oscillator basis $|n\rangle$, $n = 0, 1, \ldots$, on which D_1 and D_2 act as arbitrary Hermitian t-dependent operators, while

$$D_0 = -i\partial_t + A_0 \tag{4}$$

with A_0 a Hermitian t-dependent operator. U(1) and U(N) gauge theory are recovered as different embeddings of the noncommutative coordinates in the oscillator space. Specifically, taking the direct sum of N copies of the Fock space (which is isomorphic to a single space via $|Nn + a\rangle \sim |n, a\rangle$) we can realize the coordinates x^1, x^2 as ladder operators, i.e.,

$$(x^1 + ix^2)\,|n, a\rangle = \sqrt{2\theta n}\,|n - 1, a\rangle, \quad a = 1, \ldots, N\,. \tag{5}$$

The noncommutative partial derivative operators become

$$\partial_j = i\omega_{jk} x^k, \qquad j,k = 1,2\,, \tag{6}$$

which indeed generate translations of x^j upon commutation. Tr in (1) is understood as the trace in the full oscillator space; $2\pi\theta$ times trace over n represents integration over the noncommutative plane while the remaining trace over a represents U(N) group trace.

Using the explicit form of the operator $D_0 = -i\partial_t + A_0$ the last term of (1) is $\mathrm{Tr}(\omega A_0)$ and the action becomes

$$S = \lambda 2\pi\theta \int dt\, \mathrm{Tr}\Big(i\,\frac{2}{3} D_\mu D_\nu D_\alpha\Big)\epsilon^{\mu\nu\alpha} + 4\pi\lambda \int dt\, \mathrm{Tr}(A_0)\,. \tag{7}$$

Notice that the last term has the form of a one-dimensional CS action.

Gauge transformations act on the fields as

$$D_\mu \to U D_\mu U^{-1}\,, \tag{8}$$

where U is a t-dependent unitary transformation with the property that U acts as identity on the states $|n\rangle$ of the oscillator basis as $n \to \infty$. This property is the noncommutative version of the requirement that gauge transformations go to the identity at spatial infinity. For the level quantization argument, we consider U's which also become trivial at time infinity, that is, $U \to 1$ as $t \to \pm\infty$. In the commutative case, these requirements tell us that the maps $U:\mathbf{R}^3 \to G$ are equivalent to the maps $U:S^3 \to G$ and are classified by the winding number of the homotopy group $\Pi_3(G)$.

Under the unitary transformation (8), the change in the action (7) is given by

$$\Delta S = i4\pi\lambda \int dt\, \mathrm{Tr}(\dot{U}U^{-1}). \qquad (9)$$

Notice that with U acting as identity on states $|n\rangle$ for large n, D_μ do not change for large n, the cyclic symmetry of the trace holds, and the first term in (7) remains invariant. To see that the change of action in (9) can indeed produce a nontrivial result, consider first U's of the form

$$U|n\rangle = |n\rangle \quad \text{for } n \geq N. \qquad (10)$$

Then U is essentially a U(N) matrix and the integral $\int dt\, \mathrm{Tr}(\dot{U}U^{-1})$ is the winding number for $\Pi_1(\mathrm{U}(N))$. Specifically, we can write $U = e^{i\alpha(t)}V$ with $\det V = 1$; i.e., V is an element of SU(N). Then $\dot{V}V^{-1}$ is traceless and

$$\mathrm{Tr}(\dot{U}U^{-1}) = iN\dot{\alpha}(t). \qquad (11)$$

Since $\exp(-2\pi i/N)\mathbf{1}$ is a central element of SU(N), and $\Pi_1(\mathrm{SU}(N))$ is trivial, the periodicity of α is $2\pi/N$. With $U \to 1$ as $t \to \pm\infty$, the change in α from $t = -\infty$ to $t = +\infty$, namely, $\Delta\alpha = \alpha(\infty) - \alpha(-\infty)$, must be an integral multiple of $2\pi/N$. The change in the action (9) is given by

$$\begin{aligned} \Delta S &= 4\pi\lambda N \int dt\, \dot{\alpha}, \\ &= 4\pi\lambda N \Delta\alpha = 8\pi^2\lambda m, \end{aligned} \qquad (12)$$

where m is an integer. Setting this to be an integral multiple of 2π for single valuedness of $\exp(iS)$, we find

$$4\pi\lambda = k, \qquad (13)$$

where k is an integer. The coefficient of the CS action is thus quantized. The quantization is independent of N; the specific value of N is immaterial for the argument. The quantization is also independent of θ and conforms to the quantization of the commutative non-Abelian Chern-Simons coefficient.

The above argument applies to the U(1) theory as well as the non-Abelian theory. The difference between these theories in the commutative limit arises from the different behavior of the limit of U as θ goes to zero, as we will argue shortly. The argument is also, in essence, the argument for the level number quantization of a one-dimensional commutative CS action. The noncommutative CS action in higher dimensions also contains a term proportional to the one-dimensional CS action and hence a similar quantization of the level number holds in higher dimensions as well. Specifically, the action in $2n + 1$ dimensions is [3]

$$S = \lambda\sqrt{\det(2\pi\theta)}\, \mathrm{Tr} \sum_{k=0}^{n} \binom{n+1}{k+1} \frac{k+1}{2k+1}\, \omega^{n-k} D^{2k+1}. \qquad (14)$$

In the above, θ and ω are again the antisymmetric two-tensor and its inverse two-form specifying the noncommutativity of space, while $D = D_\mu dx^\mu$ is the operator one-form of covariant derivatives. The $k \neq n$ terms in (14) correspond to lower-dimensional Chern-Simons forms; their coefficients are chosen to reproduce the standard result upon substituting $D_\mu = -i\partial_\mu + A_\mu$. The presence of the one-dimensional term $\omega^n D \sim \sqrt{\det\omega}\, A_0$ is particularly crucial: in its absence, the equations of motion arising from (14) would admit $D_\mu = 0$ as a solution and would not reproduce ordinary noncommutative space. An argument similar to the one presented for the three-dimensional case demonstrates that all higher terms in (14) remain invariant under gauge transformations that approach the identity at infinity, while the one-dimensional term acquires a contribution proportional to the winding number of an effective U(N) transformation. A quantization of the level λ follows, in accordance with the commutative non-Abelian result.

It is instructive to demonstrate the arguments given above with an explicit example in the three-dimensional case. A concrete nontrivial unitary transformation U can be given as follows. Consider two copies of the oscillator Fock basis $|n, a\rangle$, $a = 0, 1$. We write U in the 2×2 form

$$U_{ab} = \begin{pmatrix} \sum_n A_n |n,1\rangle\langle n,1| & \sum_n B_n |n,1\rangle\langle n-1,2| \\ \sum_n B_n |n-1,2\rangle\langle n,1| & \sum_n A^*_{n+1} |n,2\rangle\langle n,2| \end{pmatrix}, \qquad (15)$$

where

$$\begin{aligned} A_n &= \frac{2\theta n - (\rho + it)^2}{2\theta n + \rho^2 + t^2}, \\ B_n &= -\frac{2i\rho\sqrt{2\theta n}}{2\theta n + \rho^2 + t^2}. \end{aligned} \qquad (16)$$

The two copies of the Fock basis may be considered as a single space with the identification

$$|n, a\rangle = |2n + a\rangle. \qquad (17)$$

In other words, we are simply splitting the oscillator Fock space into the even and odd subspaces to write U in the 2×2 form. Notice that for $n \gg \rho^2/\theta$, $U \approx \mathbf{1}$, as required. ρ is the scale size of how much this transformation differs from the identity.

We can once again calculate the integral $\int dt\, \mathrm{Tr}(\dot{U}U^{-1})$ for this configuration. Since the integral is invariant under continuous deformations, we can take very small values of ρ for this calculation. In this case, (15) gives

$$U = 1, \qquad n > 0,$$
$$= \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix}, \qquad n = 0, \tag{18}$$

where

$$e^{i\alpha(t)} = \frac{t - i\rho}{t + i\rho}. \tag{19}$$

Clearly, $\Delta\alpha = \alpha(\infty) - a(\infty) = 2\pi$ in this case, giving $\int dt\, \mathrm{Tr}(\dot{U}U^{-1}) = 2\pi i$. We have a configuration of winding number 1 of $\Pi_1(\mathrm{U}(N))$ for $N \gg \rho^2/\theta$.

As θ goes to zero, the transformation (15) becomes a winding number 1 element of $\Pi_3(\mathrm{SU}(2))$. In fact, by using coherent states, we find for $\theta \to 0$

$$U_{ab} \to \left(\frac{x^2 - \rho^2}{x^2 + \rho^2} - \frac{2i\rho\vec{x}\cdot\vec{\sigma}}{x^2 + \rho^2} \right)_{ab}, \tag{20}$$

where $x^2 = 2z\bar{z} + t^2$, $x^3 = t$, and σ_i are the Pauli matrices. As θ goes to zero, we see that U goes over to the smooth configuration of winding number 1 of $\Pi_3(\mathrm{SU}(2))$ corresponding to the stereographic map of the three-sphere.

We can also take the $\theta \to 0$ limit considering U to be a U(1)-type transformation. Embedding the states in a single Fock space as in (17), we have

$$U(t)\,|2n\rangle = A_n|2n\rangle + B_n|2n - 1\rangle,$$
$$U(t)\,|2n + 1\rangle = A^*_{n+1}|2n + 1\rangle + B_n|2n + 2\rangle. \tag{21}$$

Since $|n - 1\rangle \sim e^{i\varphi}|n\rangle$, $x^1 + ix^2 \sim z = re^{i\varphi}$, we see that $U(t)$ has a part that goes like $e^{i\varphi}$ or $e^{-i\varphi}$ for even and odd states, respectively. Similarly, it has a part that goes as $[z\bar{z} - (\rho \pm it)^2]/(z\bar{z} + \rho^2 + t^2)$ for even and odd states. Thus, considered as a single U(1)-type transformation, it has a highly oscillatory behavior on scales $\Delta r^2 \sim \theta$ and becomes singular as $\theta \to 0$. Strictly in the commutative limit, therefore, we do not need to require invariance under such transformations and there is no reason for quantization of the level number. However, if the commutative Abelian theory is viewed as the small θ-limit of the noncommutative theory, quantization persists.

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