Skew-Orthogonal Polynomials and Universality of Energy-Level Correlations

Akhilesh Pandey and Saugata Ghosh
School of Physical Sciences, Jawaharlal Nehru University, New Delhi-110067, India
(Received 22 February 2001; published 21 June 2001)

Random-matrix ensembles serve as models for quantum chaotic systems. We develop the theory of skew-orthogonal polynomials to study matrix ensembles with non-Gaussian weight functions. From the asymptotic properties of these and the orthogonal polynomials, we show that the local energy level correlations in the ensembles become universal properties independent of the global level density. This provides a rigorous justification for the universality of the Gaussian ensemble results observed in quantum chaotic systems.

DOI: 10.1103/PhysRevLett.87.024102 PACS numbers: 05.45.Mt, 02.30.–f, 03.65.Sq, 24.60.Lz

Gaussian ensembles of random matrices have been studied extensively as models for quantum chaotic systems. The three invariant Gaussian ensembles—GOE, GUE, and GSE, having invariances under orthogonal, unitary, and symplectic transformations, respectively—describe accurately the three universal patterns of energy-level correlations observed in quantum chaotic systems [1–6]. Spectra of complex nuclei, atoms, molecules, disordered mesoscopic systems, and microwave cavities provide experimental verification of the universality [1–3]. Numerical and semiclassical studies of quantum chaotic systems with few degrees of freedom as well as studies of zeros of the Riemann zeta function confirm the same [1–3]. However, except for their analytical tractability, there is no compelling physical reason to study *only* the Gaussian ensembles. In fact, the level density does not correspond to any known physical system; the universal behavior is for the local energy level statistics after suitable rescaling of the spectrum by the average spacing. It is therefore of interest [7] to study *invariant* matrix ensembles with non-Gaussian weight functions which would give very different level densities. The new ensembles have the physically relevant property [4] that the distinct matrix elements are not independent. Such ensembles are particularly useful in quantum transport studies of disordered mesoscopic systems [2,6].

In this Letter we study a wide class of non-Gaussian random-matrix ensembles with the aim to prove a conjecture of Dyson [7] that "the local statistical properties of the eigenvalues in the ensembles become universal properties independent of the global eigenvalue distributions" in the limit of large dimensionality. With this, the universality found in physical systems will have a firm justification. Proof of this would require a study of skew-orthogonal polynomials for the ensembles with orthogonal and symplectic invariances. Unlike the orthogonal polynomials (needed for the unitary case), the skew-orthogonal polynomials are defined with respect to antisymmetric scalar products and hence are much more difficult to deal with. Except for the formalism laid down by Dyson [7] and some initial results of Mehta [8], very little is known about them. Our second aim in this study is [7] "to develop the theory of skew-orthogonal polynomials until it becomes a working tool as handy as the existing theory of orthogonal polynomials."

The universal eigenvalue statistics, while independent of the weight function and location in the spectrum, will depend on the type of the invariance. Independence from the weight function was first proved by Fox and Kahn [9] for the Jacobi class of weight functions in the unitary case, while independence from the location in the spectrum ("stationarity") was proved by one of the present authors [10] for all three Gaussian ensembles. In this Letter we prove the universality rigorously for the entire class of Jacobi weights for all three types of invariant ensembles. For more general weight functions we do the same in the later part of the Letter via an ansatz for the asymptotic forms of the polynomials.

We consider matrix ensembles in which the joint-probability density of eigenvalues is given by

$$\mathcal{P}_{\beta,N}(x_1,\ldots,x_N) = c_{\beta,N}\prod_{j<k}|x_j - x_k|^\beta \prod_{i=1}^{N} w(x_i)\,, \quad (1)$$

where N is the dimensionality of the matrices, c is the normalization constant, $w(x)$ is the weight function referred to above [$w(x) = \exp(-x^2/2)$ in the Gaussian case] and the x_j are the eigenvalues. The parameter β with values $1, 2, 4$ describes, respectively, the orthogonal, unitary, and symplectic cases. (The corresponding matrix probability density is proportional to $\exp[\operatorname{tr}\log w(H)]$, where the matrix H is real, complex, or quaternion-real Hermitian for $\beta = 1, 2, 4$, respectively.) The n-level correlation function,

$$R_n^{(\beta)}(x_1,\ldots,x_n) = \frac{N!}{(N-n)!}\int dx_{n+1}\ldots\int dx_N\; \mathcal{P}_{\beta,N}(x_1,\ldots,x_N)\,, \quad (2)$$

 0031-9007/01/87(2)/024102(4)$15.00

is the probability density of observing n levels at $x_1, \ldots, x_n$ irrespective of the location of the other levels. Thus $R_1(x)$, which is normalized to N, is the level density. The local correlation functions,

$$\mathbf{R}_n^{(\beta)}(r_1, \ldots, r_n; x) = \lim_{N\to\infty} \frac{R_n^{(\beta)}(x_1, \ldots, x_n)}{R_1^{(\beta)}(x_1), \ldots, R_1^{(\beta)}(x_n)}, \quad (3)$$

describe, for large N, the level statistics in the neighborhood of x. Here, in (3), $x_j = x + r_j[R_1(x)]^{-1}$ are expressed in terms of the locally rescaled eigenvalues r_j.

Dyson [7] has shown that, for finite N, the R_n can be expressed in terms of the kernel function $S_N^{(\beta)}(x,y)$ of orthogonal or skew-orthogonal polynomials. For $\beta = 2$, the kernel is written in terms of the orthogonal polynomials $p_j(x)$:

$$S_N^{(2)}(x,y) = w(x) \sum_{j=0}^{N-1} p_j(x) p_j(y), \quad (4)$$

where the $p_j(x)$ satisfy the orthonormality relation,

$$\int p_j(x) p_k(x) w(x)\, dx = \delta_{jk}. \quad (5)$$

Similarly for $\beta = 1$ ($N =$ even), the kernel analogous to (4) is written in terms of skew-orthogonal polynomials $q_j(x)$:

$$S_N^{(1)}(x,y) = \int dz\, \epsilon(y - z) w(x) w(z) \times \sum_{m=0}^{N/2} [q_{2m}(x) q_{2m+1}(z) - q_{2m+1}(x) q_{2m}(z)], \quad (6)$$

where the $q_j(x)$ satisfy the skew-orthonormality relation,

$$\iint dx\, dy\, \epsilon(x - y) w(x) w(y) q_j(x) q_k(y) = Z_{jk}. \quad (7)$$

Here $2\epsilon(x)$ is the sign of x, and $Z_{jk} = -Z_{kj}$ has the value 1 for $k = j + 1$ with j even, the value -1 for $k = j - 1$ with j odd, and zero for $|j - k| \neq 1$. [Equations (6) and (7) can be written in simpler forms in terms of the integrated function $\psi_j(x)$ defined in (22) below.] The odd-N case of $\beta = 1$ can be handled similarly. For $\beta = 4$, we have in terms of the skew-orthogonal polynomials $t_j(x)$:

$$S_N^{(4)}(x,y) = \frac{\partial}{\partial y} \left\{ [w(x) w(y)]^{1/2} \sum_{m=0}^{N} [t_{2m}(x) t_{2m+1}(y) - t_{2m+1}(x) t_{2m}(y)] \right\}. \quad (8)$$

Here the $t_j(x)$ satisfy a different skew-orthonormality relation, viz.,

$$\int dx\, w(x) [t_j(x) t_k'(x) - t_k(x) t_j'(x)] = Z_{jk}, \quad (9)$$

which follows Mehta's definition [8], rather than the original definition of Dyson [7]. The $q_j(x)$ and $t_j(x)$, like $p_j(x)$, are polynomials of order j. The odd-order skew-orthogonal polynomials are not uniquely defined, as any multiple of the next lower even-order skew-orthogonal polynomial can be added to it.

With the above definitions, the level density is given by

$$R_1^{(\beta)}(x) = S_N^{(\beta)}(x,x), \quad (10)$$

while for $\mathbf{R_n}$ we need to consider the limit

$$S^{(\beta)}(r; x) = \lim_{N\to\infty} \frac{S_N^{(\beta)}(x, x + \Delta x)}{S_N^{(\beta)}(x,x)}, \quad (11)$$

where $r = \Delta x R_1(x)$. The detailed expressions for the R_n and $\mathbf{R_n}$ can be found in [1,7,8]. Our aim is to prove that

$$S^{(\beta)}(r; x) = \frac{\sin(\alpha \pi r)}{\alpha \pi r}, \qquad \alpha = 1 + \delta_{\beta 4}, \quad (12)$$

independent of the weight $w(x)$ and location x; with this the three types of $\mathbf{R_n}$ will become universal, identical with the Gaussian results.

We first consider the Jacobi weight function

$$w_{ab} = (1 - x)^a (1 + x)^b, \qquad |x| < 1, \quad (13)$$

with $a, b > -1$. In this case, we have for $\beta = 2$

$$p_j(x) = (h_j^{a,b})^{-1/2} P_j^{a,b}(x), \quad (14)$$

$$h_j^{a,b} = \frac{2^{a+b+1}}{(2j + a + b + 1)} \frac{\Gamma(j + a + 1)\Gamma(j + b + 1)}{\Gamma(j + 1)\Gamma(j + a + b + 1)}, \quad (15)$$

where the $P_j^{a,b}(x)$ are the Jacobi orthogonal polynomials and the $h_j^{a,b}$ are the corresponding normalizations [11]. The skew-orthogonal polynomials of the $\beta = 1$ type can be written compactly in terms of the Jacobi polynomials:

$$q_{2m}(x) = (h_{2m}^{2a+1,2b+1})^{-1/2} P_{2m}^{2a+1,2b+1}(x), \quad (16)$$

$$q_{2m+1}(x) = (h_{2m}^{2a+1,2b+1})^{-1/2} [w_{ab}(x)]^{-1} \frac{d}{dx} [w_{a+1,b+1}(x) P_{2m}^{2a+1,2b+1}(x)], \quad (17)$$

where, after the differential, q_{2m+1} turns out to be a linear combination of $P_{2m+1}^{2a+1,2b+1}$ and $P_{2m-1}^{2a+1,2b+1}$. For $\beta = 4$, the derivatives of the polynomials have compact forms:

$$t_{2m+1}'(x) = (g_{2m})^{-1/2} P_{2m}^{a,b}(x), \quad (18)$$

$$t_{2m}'(x) = (g_{2m})^{-1/2} P_{2m-1}^{a,b}(x) + \eta_{2m} t_{m-2}'(x). \quad (19)$$

To obtain $t_j(x)$, note that integral of $P_j^{a,b}$ can be written as a linear combination of $P_{j+1}^{a,b}$, $P_j^{a,b}$, $P_{j-1}^{a,b}$. The constants η_{2m} and g_{2m} are given by

$$\eta_{2m} = (g_{2m-2}/g_{2m})^{1/2} \frac{(2m + a - 1)(2m + b - 1)(4m + a + b - 5)}{(2m - 1)(2m + a + b - 1)(4m + a + b - 1)}, \tag{20}$$

$$g_{2m} = 2h_{2m}^{a,b}/(4m + a + b - 1). \tag{21}$$

To prove (16) and (17), we use an expansion of the integrated form:

$$\psi_j(x) \equiv \int \epsilon(x - y)w_{ab}(y)q_j(y)\,dy = d_j w_{a+1,b+1}(x)P_{j-1}^{2a+1,2b+1}(x) + \sum_{k=0}^{j-1} \gamma_k^{(j)}\psi_k(x), \tag{22}$$

which, on differentiation, gives a general expansion for $q_j(x)$. Note that the polynomials $P^{2a+1,2b+1}$ are orthogonal with respect to the weight $w_{2a+1,2b+1} = w_{a+1,b+1}w_{a,b}$. The skew orthogonality with lower-order polynomials gives $\gamma_k^{(j)} = 0$ for all k except $k = j - 2$; $\gamma_{j-2}^{(j)}$ can be chosen to be zero for odd j, while for even j, the nonzero $\gamma_{j-2}^{(j)}$ and the constant d_j are determined from skew orthonormality with q_{j-1} and q_{j+1}. Now differentiation of (22) gives (17) immediately while (16) follows after cancellation of the lower-order polynomials. A similar proof can be worked out in the $\beta = 4$ case. Detailed proofs, along with the results for $\beta = 1$ ($N =$ odd), will be given elsewhere [12]. We also mention that, under suitable limits, the above results can be extended to the associated Laguerre $w_a^{(L)}(x)$ and the Hermite or Gaussian $w_G(x)$ weights:

$$w_a^{(L)}(x) = x^a \exp(-x), \qquad a > -1, \tag{23}$$

$$w_G(x) = \exp(-\beta x^2/2). \tag{24}$$

The special cases $a = b = 0$ for w_{ab} (Legendre) and $a = 0$ for $w_a^{(L)}$ (Laguerre) were considered earlier by Mehta [8]. Moreover, the Gaussian results agree with those of [1,8].

Asymptotic forms (i.e., large-j forms) of the Jacobi polynomials $P_j^{a,b}(x)$ are given in [11] for finite a, b as well as the above mentioned two limiting cases. In terms of these, the asymptotic forms for $p_j(x), q_j(x), t_j(x)$ can be derived. We write them below in a form suitable for later generalization. We have

$$[w(x)]^{1/2}p_j(x) = A_j\left(\frac{2\partial\rho_j^{(2)}(x)}{\partial j}\right)^{1/2}\cos\left(\pi\int_{-\infty}^{x}\rho_j^{(2)}(x')\,dx' + \chi(x)\right), \tag{25}$$

$$w(x)q_{2m}(x) = \pi B_m(x)\rho_{2m}^{(1)}(x)\cos\left(\pi\int_{-\infty}^{x}\rho_{2m}^{(1)}(x')\,dx' + \xi(x)\right), \tag{26}$$

$$w(x)q_{2m+1}(x) = -[B_m(x)]^{-1}\left(\frac{2\partial\rho_{2m}^{(1)}(x)}{\partial(2m)}\right)\sin\left(\pi\int_{-\infty}^{x}\rho_{2m}^{(1)}(x')\,dx' + \xi(x)\right), \tag{27}$$

$$[w(x)]^{1/2}t_{2m}(x) = C_m(x)\cos\left(2\pi\int_{-\infty}^{x}\rho_m^{(4)}(x')\,dx' + \zeta(x)\right), \tag{28}$$

$$[w(x)]^{1/2}t_{2m+1}(x) = [2\pi C_m(x)\rho_m^{(4)}(x)]^{-1}\left(\frac{\partial\rho_m^{(4)}(x)}{\partial(m)}\right)\sin\left(2\pi\int_{-\infty}^{x}\rho_m^{(4)}(x')\,dx' + \zeta(x)\right). \tag{29}$$

Here $\rho_j^{(\beta)}(x)$ [$2\rho_m^{(\beta)}(x)$ for $\beta = 4$ with $j = 2m$] is the density of zeros of the polynomials, the normalization being j. [t_{2m} in (28) has additional terms giving thereby complex zeros with small imaginary parts; $2\rho_m^{(4)}$ is then density of the real parts.] $\chi(x)$, $\xi(x)$, and $\zeta(x)$ are j- and m-independent phases, while A_j, B_m, and C_m are extra factors needed in the amplitudes with $|A_j| = 1$. For the weights mentioned above $\rho_j^{(2)}$ and χ can be read off from the asymptotic forms given in [11], and then the other quantities can be derived from the above finite-m results for the polynomials. It turns out that A, B, C, χ, ξ, and ζ are not needed in the evaluation of the limit in (11). The density function $\rho_j(x)$ can be written as

$$\rho_j^{(\beta)}(x) = j\pi^{-1}(1 - x^2)^{-1/2}, \quad |x| \le 1, \tag{30}$$

$$\rho_j^{(\beta)}(x) = (\beta\pi)^{-1}x^{-1/2}(2\beta j - x)^{1/2}, \quad 0 \le x \le 2\beta j, \tag{31}$$

$$\rho_j^{(\beta)} = \pi^{-1}(2j - x^2)^{1/2}, \qquad |x| \le (2j)^{1/2}, \tag{32}$$

respectively, for Jacobi (finite a, b), associated Laguerre, and Hermite cases. To evaluate $S_N^{(\beta)}(x, y)$ for large N, we use the asymptotic forms of the polynomials in (4), (6), and (8) and replace the summation by an integral. We find from (10) that the level density R_1 approaches the density of zeros ρ_N for large N:

$$R_1^{(\beta)}(x) = \rho_N^{(\beta)}(x), \tag{33}$$

while (11) yields

$$S^{(\beta)}(r; x) = \lim_{N\to\infty}\left(\frac{\sin[\alpha\pi\Delta x R_1^{(\beta)}(x)]}{\alpha\pi\Delta x R_1^{(\beta)}(x)}\right), \tag{34}$$

giving thereby (12) in the limit. We have thus proved the universality for the entire Jacobi class of weight functions.

We now propose the ansatz that the asymptotic forms (25)–(29), proved rigorously above for the Jacobi class of weights, are valid quite generally. To make the ansatz plausible, we first write (exact) matrix-integral representations of the unnormalized polynomials:

$$p_j(x) = \left\langle \prod_{\mu=1}^{j} (x - x_\mu) \right\rangle_{2,j}, \tag{35}$$

$$q_{2m}(x) = \left\langle \prod_{\mu=1}^{2m} (x - x_\mu) \right\rangle_{1,2m}, \tag{36}$$

$$q_{2m+1}(x) = \left\langle \left(x + \sum_{\mu=1}^{2m} x_\mu \right) \prod_{\nu=1}^{2m} (x - x_\nu) \right\rangle_{1,2m}, \tag{37}$$

$$t_{2m}(x) = \left\langle \prod_{\mu=1}^{m} (x - x_\mu)^2 \right\rangle_{4,m}, \tag{38}$$

$$t_{2m+1}(x) = \left\langle \left(x + 2 \sum_{\mu=1}^{2m} x_\mu \right) \prod_{\nu=1}^{m} (x - x_\nu)^2 \right\rangle_{4,m}, \tag{39}$$

where $\langle F(x_1, \ldots, x_N) \rangle_{\beta,N}$ is the average of F with respect to the joint-probability density $\mathcal{P}_{\beta,N}$ of (1). The $\beta = 2$ result (35) is due to Eynard [13]; proof of the skew-orthogonal forms (36)–(39) can be worked out similarly [12,14]. Now, ignoring the correlations in the x_μ, we see from (35)–(39) that, asymptotically, the zeros of the polynomials have the same density as that of $\langle x_\mu \rangle$, giving thereby the relation (33). By considering the spacing between the consecutive zeros, we can verify that the phases in (25)–(29) are all consistent with the definition of ρ_j. Using (5), (7), (9), and (10), one can establish the form of the amplitudes. Thus the ansatz gives (34) and hence the universal result (12) for general weights.

We mention finally that $R_1(x)$, and hence $\rho(x)$ in (25)–(29), can be derived independently. For this we note first that (1) and (2) give

$$\frac{\partial R_1^{(\beta)}(x)}{\partial x} = \beta \int \frac{R_2^{(\beta)}(x, y)}{x - y} \, dy + \frac{w'(x)}{w(x)} R_1^{(\beta)}(x), \tag{40}$$

which, for large N, becomes [7,15–17]

$$\beta R_1^{(\beta)}(x) \int \frac{R_1^{(\beta)}(y)}{x - y} \, dy + \frac{w'(x)}{w(x)} R_1^{(\beta)}(x) = 0, \tag{41}$$

where the principal value of the integral is used. Using the resolvent [18],

$$G(z) = \int \frac{R_1(y)}{z - y} \, dy, \tag{42}$$

one can solve (41) for many weight functions. For example, with careful consideration of the singularities in $w'(x)/w(x)$, we find (30)–(32) for the Jacobi class of weights. This, then, gives the essential parts of the asymptotic forms (25)–(29) more explicitly. Applying this to $w(x) = \exp[-NV(x)]$, where $V(x)$ is a low-order polynomial, we get, from (25), the Brezin and Zee [19] ansatz; (26)–(29) may then be considered as the skew-orthogonal extensions of the Brezin and Zee ansatz.

We have thus established the universality and stationarity of local statistical properties of energy levels rigorously for the Jacobi class of weight functions and via an ansatz for more general weight functions. The matrix-integral representations of the polynomials—orthogonal as well as skew-orthogonal ones—appear to be promising for rigorous studies with general weight functions. We have also given a formalism for deriving the (nonuniversal) level density without using the polynomial method.

The concept of skew orthogonality may turn out to be useful for other random-matrix systems and in other contexts. It seems [15] that the local correlation functions have universal properties also for the Brownian-motion matrix ensembles [7], viz., ensembles interpolating between the invariant ones. These ensembles are useful in the studies of small symmetry breaking in quantum chaotic systems [16]. In the GOE-GUE [20] and GSE-GUE [21] interpolations, the concept of skew orthogonality has been implicitly used. It remains to be seen whether the skew-orthogonal polynomials will play any role in the more general Brownian-motion ensembles. From the asymptotic forms it also seems tempting to look for new methods of semiclassical quantization [22] of chaotic systems using skew-orthogonal functions as tools.

[1] M. L. Mehta, *Random Matrices* (Academic, New York, 1991).
[2] T. Guhr, A. M. Groeling, and H. A. Weidenmueller, Phys. Rep. **299**, 189 (1998).
[3] O. Bohigas and M. J. Giannoni, *Lecture Notes in Physics* (Springer, Berlin, 1984), Vol. 209.
[4] T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, Rev. Mod. Phys. **53**, 385 (1981).
[5] F. Haake, *Quantum Signatures of Chaos* (Springer, Berlin, 2001).
[6] C. W. J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997).
[7] F. J. Dyson, J. Math. Phys. (N.Y.) **13**, 90 (1972).
[8] M. L. Mehta, *Matrix Theory* (Hindustan Publishing Corporation, Delhi, 1989).
[9] D. Fox and P. B. Kahn, Phys. Rev. **134**, B1151 (1964).
[10] A. Pandey, Ann. Phys. (N.Y.) **119**, 170 (1979).
[11] G. Szego, *Orthogonal Polynomials* (American Mathematical Society, Providence, 1939).
[12] S. Ghosh and A. Pandey (to be published).
[13] B. Eynard, Nucl. Phys. **B506**, 633 (1997).
[14] B. Eynard, Saclay Report No. SPT-00/170, CRM 2698.
[15] A. Pandey, Chaos Solitons Fractals **5**, 1275 (1995).
[16] J. B. French, V. K. B. Kota, A. Pandey, and S. Tomsovic, Ann. Phys. (N.Y.) **181**, 198 (1988).
[17] A. Pandey and P. Shukla, J. Phys. A **24**, 3907 (1991).
[18] A. Pandey, Ann. Phys. (N.Y.) **134**, 110 (1981).
[19] E. Brezin and A. Zee, Nucl. Phys. **B402**, 613 (1993).
[20] A. Pandey and M. L. Mehta, Commun. Math. Phys. **87**, 449 (1983).
[21] M. L. Mehta and A. Pandey, J. Phys. A **16**, 2655 (1983).
[22] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, New York, 1990).